1. Prove that $\sin(n+1)A \sin(n+2)A + \cos(n+1)A \cos(n+2)A = \cos A$

Soln. We know that

$$\cos(\theta - \phi) = \cos\theta \cos\theta + \sin\theta \sin\phi$$

Here let $(n+1)A = \phi$, $(n+2)A = \theta$

Applying, we have

$$\sin\phi \sin\theta + \cos\phi \cos\theta = \cos(\theta - \phi)$$

$$= \cos[(n+2)A - (n+1)A]$$

$$= \cos A \quad \text{Proved.}$$

2. If $(A+B) = \frac{\pi}{4}$, prove that $(1+\tan A)(1+\tan B) = 2$

Soln. $\because A+B = \frac{\pi}{4}$

$\therefore \tan(A+B) = \tan\frac{\pi}{4}$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$$

$$\Rightarrow \tan A + \tan B = 1 - \tan A \tan B.$$

$$\Rightarrow \tan A + \tan B + \tan A \tan B = 1 \quad \text{—— (1)}$$

Now, we have $(1+\tan A)(1+\tan B)$

$$= 1 + \tan A + \tan B + \tan A \tan B$$

$$= 1 + 1 = 2 \quad \text{Ans. using (1)}$$

3. Prove that

$$\sin^2\left(\frac{\pi}{8}+\frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right) = \frac{1}{\sqrt{2}}\sin A$$

Soln- We know that

$$\sin^2 x - \sin^2 y = \sin(x+y)\sin(x-y)$$

$$\therefore \sin^2\left(\frac{\pi}{8}+\frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right)$$

$$= \sin\left(\frac{\pi}{8}+\cancel{\frac{A}{2}}+\frac{\pi}{8}-\cancel{\frac{A}{2}}\right)\sin\left(\cancel{\frac{\pi}{8}}+\frac{A}{2}-\cancel{\frac{\pi}{8}}+\frac{A}{2}\right)$$

$$= \sin\frac{\pi}{4}\sin A \quad = \frac{1}{\sqrt{2}}\sin A$$

4. Prove that $\tan 70^\circ = \tan 20^\circ + 2\tan 50^\circ$

Soln. we know $\tan(A+B) = \dfrac{\tan A + \tan B}{1-\tan A \tan B}$.

$$\therefore \tan 70^\circ = \tan(50^\circ+20^\circ)$$

$$= \frac{\tan 50^\circ + \tan 20^\circ}{1-\tan 50^\circ \tan 20^\circ}$$

$$\Rightarrow \tan 70^\circ(1-\tan 50^\circ \tan 20^\circ) = \tan 50^\circ + \tan 20^\circ$$

$$\Rightarrow \tan 70^\circ - \tan 50^\circ \tan 20^\circ \tan 70^\circ = \tan 50^\circ + \tan 20^\circ$$

$$\Rightarrow \tan 70^\circ - \tan 50^\circ \tan 20^\circ \cot 20^\circ = \tan 50^\circ + \tan 20^\circ$$

$$\Rightarrow \tan 70^\circ - \tan 50^\circ = \tan 50^\circ + \tan 20^\circ$$

$$\Rightarrow \tan 70^\circ = 2\tan 50^\circ + \tan 20^\circ$$

$$\left[\because \cot 20^\circ = \tan 70^\circ,\quad \tan\theta\cot\theta = 1\right]$$

5 If $\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha)=-\frac{3}{2}$

Prove that $\cos\alpha+\cos\beta+\cos\gamma=\sin\alpha+\sin\beta+\sin\gamma=0$

Soln: $2\cos(\alpha-\beta)+2\cos(\beta-\gamma)+2\cos(\gamma-\alpha)=-3$

$$2(\cos\alpha\cos\beta+\sin\alpha\sin\beta)+2(\cos\beta\cos\gamma+\sin\beta\sin\gamma)$$
$$+2(\cos\gamma\cos\alpha+\sin\gamma\sin\alpha)+3=0$$

$$\Rightarrow 2\cos\alpha\cos\beta+2\cos\beta\cos\gamma+2\cos\gamma\cos\alpha+2\sin\alpha\sin\beta$$
$$+2\sin\beta\sin\gamma+2\sin\gamma\sin\alpha+\sin^2\alpha+\cos^2\alpha+\sin^2\beta+\cos^2\beta$$
$$+\sin^2\gamma+\cos^2\gamma=0$$

$$\Rightarrow \sin^2\alpha+\sin^2\beta+\sin^2\gamma+2\sin\alpha\sin\beta+2\sin\beta\sin\gamma$$
$$+2\sin\gamma\sin\alpha+\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta$$
$$+2\cos\beta\cos\gamma+2\cos\gamma\cos\alpha=0$$

$$\Rightarrow(\sin\alpha+\sin\beta+\sin\gamma)^2+(\cos\alpha+\cos\beta+\cos\gamma)^2=0$$

$$\Rightarrow \sin\alpha+\sin\beta+\sin\gamma=0 \text{ and } \cos\alpha+\cos\beta+\cos\gamma=0$$

[As the sum of two positive numbers can not be zero)

This is possible if each part is equal to zero.

6. If $a\tan\alpha+b\tan\beta=(a+b)\tan\left(\frac{\alpha+\beta}{2}\right)$

where $\alpha\neq\beta$, prove that $a\cos\beta=b\cos\alpha$

Soln: $$a\tan\alpha+b\tan\beta=a\tan\frac{\alpha+\beta}{2}+b\tan\frac{\alpha+\beta}{2}$$

$$\Rightarrow a\left[\tan\alpha-\tan\frac{\alpha+\beta}{2}\right]=b\left[\tan\left(\frac{\alpha+\beta}{2}\right)-\tan\beta\right]$$

$$\Rightarrow a\left[\frac{\sin\alpha}{\cos\alpha}-\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha+\beta}{2}}\right]=b\left[\frac{\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}-\frac{\sin\beta}{\cos\beta}\right]$$

$$\Rightarrow a\,\frac{\sin\alpha\cos\left(\frac{\alpha+\beta}{2}\right)-\cos\alpha\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\alpha\;\cancel{\cos\left(\frac{\alpha+\beta}{2}\right)}}=b\,\frac{\sin\left(\frac{\alpha+\beta}{2}\right)\cos\beta-\cos\left(\frac{\alpha+\beta}{2}\right)\sin\beta}{\cancel{\cos\frac{\alpha+\beta}{2}}\;\cos\beta}$$

$$\Rightarrow a\,\frac{\sin\left(\alpha-\frac{\alpha+\beta}{2}\right)}{\cos\alpha}=b\,\frac{\sin\left(\frac{\alpha+\beta}{2}-\beta\right)}{\cos\beta}.$$

$$\Rightarrow a\,\frac{\cancel{\sin\left(\frac{\alpha-\beta}{2}\right)}}{\cos\alpha}=\frac{b\;\cancel{\sin\left(\frac{\alpha-\beta}{2}\right)}}{\cos\beta}\Rightarrow a\cos\beta=b\cos\alpha$$

7 Prove that

$$\cos2\alpha\cos2\beta+\sin^2(\alpha-\beta)-\sin^2(\alpha+\beta)=\cos2(\alpha+\beta)$$

Soln. $\cos2\alpha\cos2\beta+\sin(\alpha-\beta+\alpha+\beta)\sin(\alpha-\beta-\alpha-\beta)$

$[\because \sin^2A-\sin^2B=\sin(A+B)\sin(A-B)]$

$\Rightarrow \cos2\alpha\cos2\beta+\sin2\alpha\sin(-2\beta)$

$\Rightarrow \cos2\alpha\cos2\beta-\sin2\alpha\sin2\beta$ $\quad[\because \sin(-\theta)=-\sin\theta]$

$\Rightarrow \cos(2\alpha+2\beta)=\cos2(\alpha+\beta)$

$[\because \cos A\cos B-\sin A\sin B=\cos(A+B)]$

8. If $\tan(\pi\cos\theta)=\cot(\pi\sin\theta)$ prove that

$$\cos\left(\theta-\frac{\pi}{4}\right)=\frac{1}{2\sqrt{2}}$$

Soln. $\tan(\pi\cos\theta)=\tan\left(\frac{\pi}{2}-\pi\sin\theta\right)$

$\Rightarrow \pi\cos\theta=\frac{\pi}{2}-\pi\sin\theta$

$\Rightarrow \cos\theta = \frac{1}{2} - \sin\theta \Rightarrow \cos\theta + \sin\theta = \frac{1}{2}$

$\Rightarrow \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta = \frac{1}{2\sqrt{2}}$

$\Rightarrow \cos\theta\cos\frac{\pi}{4} + \sin\theta\sin\frac{\pi}{4} = \frac{1}{2\sqrt{2}}$

$\Rightarrow \cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$

9. If $\sin\alpha + \sin\beta = a$ and $\cos\alpha + \cos\beta = b$

Show that (i) $\cos(\alpha+\beta) = \frac{b^2 - a^2}{b^2 + a^2}$

(ii) $\sin(\alpha+\beta) = \frac{2ab}{a^2+b^2}$

Soln.

$\because \frac{a}{b} = \frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta} = \frac{\cancel{2}\sin\frac{\alpha+\beta}{2}\cancel{\cos\frac{\alpha-\beta}{2}}}{\cancel{2}\cos\frac{\alpha+\beta}{2}\cancel{\cos\frac{\alpha-\beta}{2}}} = \tan\frac{\alpha+\beta}{2}$

$\left[\because \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}, \ \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right]$

$\therefore \cos(\alpha+\beta) = \frac{1-\tan^2\frac{\alpha+\beta}{2}}{1+\tan^2\frac{\alpha+\beta}{2}} = \frac{1-\left(\frac{a}{b}\right)^2}{1+\left(\frac{a}{b}\right)^2} = \frac{b^2-a^2}{b^2+a^2}$

$\sin(\alpha+\beta) = \frac{2\tan\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\frac{\alpha+\beta}{2}} = \frac{2\,\frac{a}{b}}{1+\left(\frac{a}{b}\right)^2} = \frac{\frac{2a}{b}}{\frac{b^2+a^2}{b^2}}$

$= \frac{2a}{b} \times \frac{b^2}{b^2+a^2} = \frac{2ab}{b^2+a^2}$

10. If α and β are the solution of $a\cos\theta + b\sin\theta = c$

Then show that (i) $\cos(\alpha+\beta) = \frac{a^2-b^2}{a^2+b^2}$

(ii) $\cos(\alpha-\beta) = \frac{2c^2-(a^2+b^2)}{a^2+b^2}$

Soln: $\because$ α and β are the solution of $a\cos\theta + b\sin\theta = c$

$\therefore$ $a\cos\alpha + b\sin\alpha = c$ —— (i)

$a\cos\beta + b\sin\beta = c$ —— (ii)

$\therefore$ $a(\cos\alpha - \cos\beta) + b(\sin\alpha - \sin\beta) = 0$

$a\left(-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\right) + b\left(2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\right) = 0$

$\cancel{2}a\sin\frac{\alpha+\beta}{2}\cancel{\sin\frac{\alpha-\beta}{2}} = \cancel{2}b\cos\frac{\alpha+\beta}{2}\cancel{\sin\frac{\alpha-\beta}{2}}$

$(\because \alpha \neq \beta)$

$\therefore$ $\tan\frac{\alpha+\beta}{2} = \frac{b}{a}$

$\therefore$ $\cos(\alpha+\beta) = \dfrac{1-\tan^2\frac{\alpha+\beta}{2}}{1+\tan^2\frac{\alpha+\beta}{2}}$ $\left[\because \cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right]$

$= \dfrac{1-\left(\frac{b}{a}\right)^2}{1+\left(\frac{b}{a}\right)^2} = \dfrac{a^2-b^2}{a^2+b^2}$

(ii) Squaring and adding (i) and (ii), we get

$2c^2 = a^2\cos^2\alpha + b^2\sin^2\alpha + 2ab\sin\alpha\cos\alpha$
$+ a^2\cos^2\beta + b^2\sin^2\beta + 2ab\sin\beta\cos\beta$

$= a^2(1-\sin^2\alpha) + b^2\sin^2\alpha + ab\sin 2\alpha$
$+ a^2\cos^2\beta + b^2(1-\cos^2\beta) + ab\sin 2\beta$

$2c^2 = a^2 - a^2\sin^2\alpha + b^2\sin^2\alpha + a^2\cos^2\beta + b^2 -$
$b^2\cos^2\beta + ab(\sin 2\alpha + \sin 2\beta)$

$\Rightarrow 2c^2 - (a^2+b^2) = ab(\sin 2\alpha + \sin 2\beta) + (b^2-a^2)\sin^2\alpha$
$+ (a^2-b^2)\cos^2\beta.$

$$= ab(\sin 2\alpha + \sin 2\beta) + (a^2-b^2)[\cos^2\beta - \sin^2\alpha]$$

$$\Rightarrow 2c^2 - (a^2+b^2) = ab\, 2\sin(\alpha+\beta)\cos(\alpha-\beta) + (a^2-b^2)\cos(\beta+\alpha)\cos(\beta-\alpha)$$

$$= \cos(\alpha-\beta)\left[2ab\sin(\alpha+\beta) + (a^2-b^2)\cos(\alpha+\beta)\right]$$

$$= \cos(\alpha-\beta)\left[2ab\frac{2\tan\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\frac{\alpha+\beta}{2}} + (a^2-b^2)\frac{(a^2-b^2)}{a^2+b^2}\right]$$

$$= \cos(\alpha-\beta)\left[\frac{2ab \times 2\frac{b}{a}}{1+\left(\frac{b}{a}\right)^2} + \frac{(a^2-b^2)^2}{a^2+b^2}\right]$$

$$= \cos(\alpha-\beta)\left[\frac{4b^2a^2}{a^2+b^2} + \frac{(a^2-b^2)^2}{a^2+b^2}\right]$$

$$= \cos(\alpha-\beta)\left[\frac{(a^2+b^2)^{\cancel{2}}}{\cancel{(a^2+b^2)}}\right]$$

$$\therefore \cos(\alpha-\beta) = \frac{2c^2-(a^2+b^2)}{a^2+b^2}$$

11. If an angle α be divided into two part such that the tangent of one part is λ times the tangent of the other part, Then prove that their difference θ is given by $\sin\theta = \frac{\lambda-1}{\lambda+1}\sin\alpha$

Soln: Since angle differ by magnitude θ, Hence, let one angle is β Then other angle will be $\alpha-\beta$

$\therefore \quad \theta = \alpha - B - B = \alpha - 2\beta$

$\because$ According to question, $\tan\beta = \lambda \tan(\alpha - B)$

$$\Rightarrow \quad \lambda = \frac{\tan B}{\tan(\alpha - B)}$$

using componendo and dividendo, we get

$$\frac{\lambda+1}{\lambda-1} = \frac{\tan B + \tan(\alpha-B)}{\tan B - \tan(\alpha-B)}$$

$$= \frac{\dfrac{\sin B}{\cos B} + \dfrac{\sin(\alpha-B)}{\cos(\alpha-B)}}{\dfrac{\sin B}{\cos B} - \dfrac{\sin(\alpha-B)}{\cos(\alpha-B)}}$$

$$= \frac{\sin B\cos(\alpha-B) + \cos B\sin(\alpha-B)}{\sin B\cos(\alpha-B) - \cos B\sin(\alpha-B)}$$

$$= \frac{\sin(\cancel{B} + \alpha - \cancel{B})}{\sin\{B - (\alpha - B)\}}$$

$$\frac{\lambda+1}{\lambda-1} = \frac{\sin\alpha}{\sin\theta}$$

$\because B - (\alpha - B) = \theta$ diff of angles.

$$\therefore \quad \sin\theta = \frac{\lambda-1}{\lambda+1}\sin\alpha$$

12. Evaluate:

$$\frac{1}{\sin\theta\sin2\theta} + \frac{1}{\sin2\theta\sin3\theta} + \frac{1}{\sin3\theta\sin4\theta} + \cdots\cdots$$

$+ n$ terms

Soln. $\frac{1}{\sin\theta}\left[\frac{\sin\theta}{\sin\theta\sin 2\theta}+\frac{\sin\theta}{\sin 2\theta\sin 3\theta}+\frac{\sin\theta}{\sin 3\theta\sin 4\theta}+\cdots\right]$

$$=\frac{1}{\sin\theta}\left[\frac{\sin(2\theta-\theta)}{\sin\theta\sin 2\theta}+\frac{\sin(3\theta-2\theta)}{\sin 3\theta\sin 2\theta}+\frac{\sin(4\theta-3\theta)}{\sin 4\theta\sin 3\theta}+\cdots\right]$$

$$=\frac{1}{\sin\theta}\left[\frac{\cancel{\sin 2\theta}\cos\theta}{\sin\theta\cancel{\sin 2\theta}}-\frac{\cos 2\theta\cancel{\sin\theta}}{\cancel{\sin\theta}\sin 2\theta}+\frac{\cancel{\sin 3\theta}\cos 2\theta}{\cancel{\sin 3\theta}\sin 2\theta}-\frac{\cos 3\theta\cancel{\sin 2\theta}}{\sin 3\theta\cancel{\sin 2\theta}}\right.$$

$$=\frac{1}{\sin\theta}\left[\cot\theta-\cancel{\cot 2\theta}+\cancel{\cot 2\theta}-\cancel{\cot 3\theta}+\cancel{\cdots}+\cancel{\cot n\theta}-\cot(n+1)\theta\right]$$

$$=\frac{1}{\sin\theta}\left[\cot\theta-\cot(n+1)\theta\right]\ \text{Ans.}$$

13. Evaluate

$\tan\theta\sec 2\theta+\tan 2\theta\sec 4\theta+\tan 4\theta\sec 8\theta+\cdots+n$ terms

Soln. $\tan\theta\sec 2\theta=\frac{\sin\theta}{\cos\theta}\times\frac{1}{\cos 2\theta}=\frac{\sin(2\theta-\theta)}{\cos\theta\cos 2\theta}$

$$=\frac{\sin 2\theta\cancel{\cos\theta}}{\cancel{\cos\theta}\cos 2\theta}-\frac{\cancel{\cos 2\theta}\sin\theta}{\cos\theta\cancel{\cos 2\theta}}$$

$$=\tan 2\theta-\tan\theta$$

Similarly, $\tan 2\theta\sec 4\theta=\cancel{\tan 4\theta}-\cancel{\tan 2\theta}$

$\tan 4\theta\sec 8\theta=\cancel{\tan 8\theta}-\cancel{\tan 4\theta}$

$- - -\qquad - \cancel{-} \quad - \quad \cancel{-}$

$\tan 2^{n-1}\theta\sec 2^{n}\theta=\tan 2^{n}\theta-\cancel{\tan 2^{n-1}\theta}$

Adding all above, we get-

$\tan\theta \sec 2\theta + \tan 2\theta \sec 4\theta + \cdots + n \text{ terms} = \tan 2^n\theta - \tan\theta$

14. Prove that $\sin\theta \sec 3\theta = \frac{1}{2}(\tan 3\theta - \tan\theta)$ and hence find the sum to n terms of the series

$\sin\theta \sec 3\theta + \sin 3\theta \sec 3^2\theta + \sin 3^2\theta \sec 3^3\theta + \cdots$

Sol^n:

$$\sin\theta \sec 3\theta = \frac{\sin\theta}{\cos 3\theta} = \frac{2\sin\theta\cos\theta}{2\cos 3\theta\cos\theta}$$

$$= \frac{\sin 2\theta}{2\cos 3\theta\cos\theta} = \frac{1}{2}\left[\frac{\sin(3\theta-\theta)}{\cos 3\theta\cos\theta}\right]$$

$$= \frac{1}{2}\left[\frac{\sin 3\theta\cancel{\cos\theta}}{\cos 3\theta\cancel{\cos\theta}} - \frac{\cancel{\cos 3\theta}\sin\theta}{\cancel{\cos 3\theta}\cos\theta}\right]$$

$$= \frac{1}{2}\left[\tan 3\theta - \tan\theta\right]$$

Now,

$\sin\theta \sec 3\theta + \sin 3\theta \sec 3^2\theta + \sin 3^2\theta \sec 3^3\theta + \cdots$

$$= \frac{1}{2}\left[\cancel{\tan 3\theta} - \tan\theta\right] + \frac{1}{2}\left[\cancel{\tan 3^2\theta} - \cancel{\tan 3\theta}\right] + \frac{1}{2}\left[\cancel{\tan 3^3\theta} - \cancel{\tan 3^2\theta}\right]$$
$$+ \cdots + \frac{1}{2}\left[\tan 3^n\theta - \cancel{\tan 3^{n-1}\theta}\right]$$

$$= \frac{1}{2}\left[\tan 3^n\theta - \tan\theta\right] \text{ Ans.}$$

15. Evaluate

$\tan x \tan 2x + \tan 2x \tan 3x + \cdots + \tan nx \tan(n+1)x$

Soln. we know $\tan x = \tan[(n+1) - (nx)]$

$$= \frac{\tan(n+1)x - \tan nx}{1 + \tan(n+1)x \tan nx}$$

$$\Rightarrow \quad 1 + \tan(n+1)x \tan x = \frac{\tan(n+1)x - \tan nx}{\tan x}$$

$$\Rightarrow \quad \tan(n+1)x \tan x = \frac{\tan(n+1)x - \tan nx}{\tan x} - 1$$

$\therefore$ putting $n = 1, 2, 3 \ldots$

$$\tan 2x \tan x = \frac{\tan 2x - \tan x}{\tan x} - 1$$

$$\tan 3x \tan 2x = \frac{\tan 3x - \tan 2x}{\tan x} - 1$$

$$\tan 4x \tan 3x = \frac{\tan 4x - \tan 3x}{\tan x} - 1$$

$-\quad -\quad -$

$$\tan(n+1)x \tan nx = \frac{\tan(n+1)x - \tan nx}{\tan x} - 1$$

Adding all above we get,

$$\tan 2x \tan x + \tan 3x \tan 2x + \cdots + \tan(n+1)x \tan nx$$

$$= \frac{\tan(n+1)x - \tan x}{\tan x} - n$$

$$= \frac{\tan(n+1)x}{\tan x} - (1+n)$$