Differentiation

CONCEPT NOTES

Differentiation

Now that we have seen the concepts of continuity and differentiability, we will start developing applications of these concepts. These applications include differentiation (developing techniques and formulae to differentiate a given function), maxima and minima, drawing tangents and normals (to some given curve at some given point), studying monotonicity of functions, and so on. In this chapter, we will study differentiation which is just an extension of the differentiability concept (in fact, we already have studied almost everything pertaining to differentiation). Our purpose here is to find out the derivatives of some standard functions by first principles, and then use these results whenever we want to differentiate any arbitrary given function.

Section - 1 INTRODUCTION TO DIFFERENTIATION

In the previous chapter, we saw in great detail the meaning of evaluating the derivative (differentiating) of a given function at any given point.

We summarize that discussion briefly here:

If $f(x)$ is a differentiable function for a given x, this means that we can draw a unique tangent to $f(x)$ for that given x. The slope of this unique tangent is called the derivative of $f(x)$ for that given x. *The process of finding the derivative is known as differentiation*.

For example, for $f(x) = x^2$, the derivative at any given x has the value $2x$ (we evaluated this by first principles in the previous chapter.) This means that the slope of the tangent drawn to $f(x)$ at any given x has the numerical value $2x$.

Equivalently stated, we can differentiate $f(x) = x^2$ to get $f'(x) = 2x$.

$\{f'(x)$ represents the derivative of $f(x)\}$.

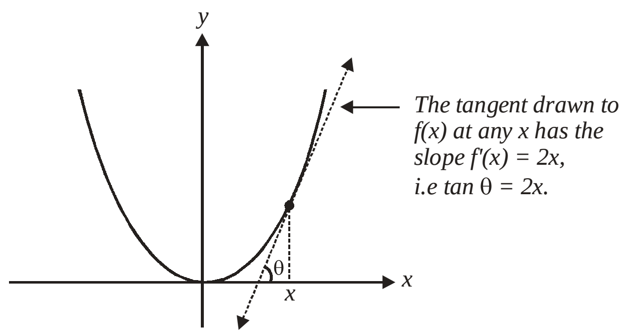

Fig - 1

Recall that we can differentiate a function at a given point only if the LHD and RHD at that point have equal values. If they do, then

$$f'(x)=\frac{d\left(f(x)\right)}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f(x-h)-f(x)}{-h}$$

The notation that we use to signify the derivative of $y=f(x)$ is either $f'(x)(\text{or } y')$ or $\frac{d f(x)}{dx}\left(\text{or } \frac{dy}{dx}\right)$

We need to understand here the significance of the notation $\frac{dy}{dx}$ (the derivative of y with respect to the variable x)

Recall that to evaluate the derivative (slope of the tangent) of $y = f(x)$ at a given point, we first drew a secant passing through that point and then let that secant tend to a tangent as follows:

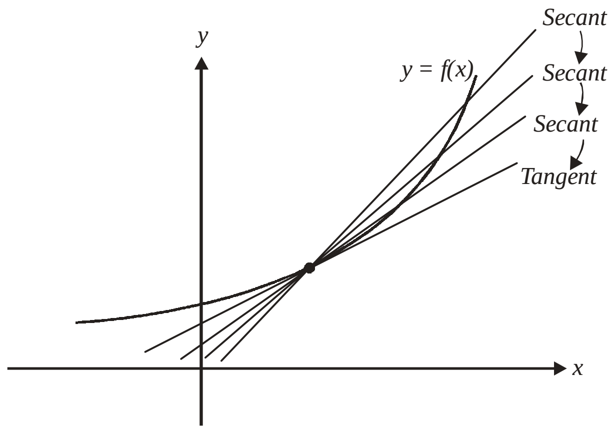

Fig - 2

The slope of any secant can be written easily:

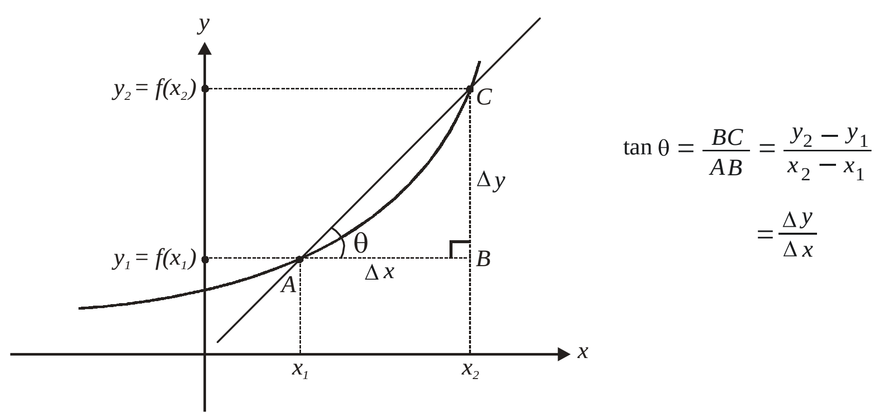

Fig - 3

The slope is obvious from the figure:

$$\tan\theta=\frac{BC}{AB}=\frac{y_2-y_1}{x_2-x_1}=\frac{\Delta y}{\Delta x}$$

Now, to evaluate the derivative at x_1, we need to make the secant AC tend to a tangent at A by letting x_2 approach

$x_1 \left(x_2 \to x_1\right)$ or equivalently, by letting $\Delta x \to 0$.

As Δx becomes an infinitesmally small quantity (approaches 0), the corresponding Δy will also become infinitesmally small (will approach 0), but the ratio $\frac{\Delta y}{\Delta x}$ will become an increasingly accurate representation of the slope of the tangent at *A*.

An infinitesmally small change in the *x* value is represented by *dx* instead of Δx.

Similarly, an infinitesmally small change in the *y* value would be represented by *dy* instead of Δy.

Therefore,

$$\lim_{x_2 \to x_1} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = y'$$

You should now be clear about the notation $\frac{dy}{dx}$. We will use $\frac{dy}{dx}, \frac{d f(x)}{dx}, f'(x)$ or y' interchangeably to represent the derivative of $y = f(x)$ at any given *x*.

Note: From now on you should always keep in mind that *d*(*variable*) represents an infinitesmally small change in the variable value while Δ (*variable*) represents a finite change in the variable value.

Section - 2 DIFFERENTIATION OF STANDARD FUNCTIONS

By now, the meaning and geometrical significance of differentiation should be pretty clear to you. We will use this knowledge to evaluate the derivatives of some standard functions in this section.

You will notice that while differentiating these functions, we will only use the expression for the RHD; we could equivalently use the LHD also since all the functions we will be concerned with in this section are differentiable (except at discontinuous points); that the LHD and RHD are equal for each of these functions at a given *x* can be easily verified.

1. $\boxed{f(x) = k}$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{k-k}{h} = 0$$

This is intuitively true also since the graph for a constant function is a horizontal line.

2. $\boxed{f(x) = x}$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h) - x}{h} = 1$$

This corresponds to the fact that the line $f(x) = x$ is inclined at 45° to the *x*–axis (and tan 45° is 1).

3. $\boxed{f(x) = mx + c}$:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{m(x+h)+c-(mx+c)}{h} = m$$

This is again a straight forward result: 'm' is the slope of $f(x) = mx + c$ so it must equal $f'(x)$.

4. $\boxed{f(x) = x^2}$:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0}\frac{h^2 + 2xh}{h} = 2x$$

We have already obtained this result earlier.

5. $\boxed{f(x) = x^n}$:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0}\left\{\frac{x^n\left(1+\frac{h}{x}\right)^n - x^n}{h}\right\}$$

$$= x^n \lim_{h\to 0}\left\{\frac{\left(1+\frac{h}{x}\right)^n - 1}{h}\right\}$$

$$= x^n \lim_{h\to 0}\frac{\left\{1+\frac{nh}{x}+n\frac{(n-1)}{2!}\frac{h^2}{x^2}+....\right\}-1}{h}$$ [By the binomial expansion]

$$= x^n \cdot \frac{n}{x} = nx^{n-1}$$

So, for example, $\frac{d(x^2)}{dx} = 2.x^{2-1} = 2x$ and $\frac{d(x^3)}{dx} = 3.x^{3-1} = 3x^2$ and so on.

6. $\boxed{f(x)=\sin x}$:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$=\lim_{h\to 0}\frac{2\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$=\lim_{h\to 0}\left\{\cos\left(x+\frac{h}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}$$

$= \cos x.$

7. $\boxed{f(x)=\cos x}$:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$=\lim_{h\to 0}\left\{\frac{-2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h}\right\}$$

$$=\lim_{h\to 0}\left\{-\sin\left(x+\frac{h}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}$$

$= -\sin x.$

8. $\boxed{f(x)=\tan x}$:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h}$$

$$=\lim_{h\to 0}\left[\frac{\tan h}{h}\cdot\{1+\tan x\cdot\tan(x+h)\}\right]$$

$= 1 + \tan^2 x$

$= \sec^2 x.$

9. $\boxed{f(x) = \sec x}$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sec(x+h) - \sec x}{h}$$

$$= \lim_{h \to 0} \frac{\cos x - \cos(x+h)}{h \cos x \cos(x+h)}$$

$$= \lim_{h \to 0} \left\{ \frac{2 \sin\left(x + \frac{h}{2}\right) \sin \frac{h}{2}}{h \cos x \cos(x+h)} \right\}$$

$$= \lim_{h \to 0} \left\{ \frac{\sin\left(x + \frac{h}{2}\right)}{\cos x \cos(x+h)} \left(\frac{\sin \frac{h}{2}}{\frac{h}{2}} \right) \right\}$$

$= \sec x \tan x$.

10. $\boxed{f(x) = \text{cosec } x}$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\text{cosec}(x+h) - \text{cosec } x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x - \sin(x+h)}{h \sin x \sin(x+h)}$$

$$= \lim_{h \to 0} \left\{ \frac{-2 \cos\left(x + \frac{h}{2}\right) \sin \frac{h}{2}}{h \sin x \sin(x+h)} \right\}$$

$$= \lim_{h \to 0} \left\{ \frac{-\cos\left(x + \frac{h}{2}\right)}{\sin x \sin(x+h)} \left(\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right) \right\}$$

$= -\text{cosec } x \cot x$

11. $\boxed{f(x)=\cot x}$:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\cot(x+h)-\cot x}{h}$$

$$=\lim_{h\to 0}\frac{\sin x\cos(x+h)-\cos x\sin(x+h)}{h\sin x\sin(x+h)}$$

$$=\lim_{h\to 0}\frac{-\sin h}{h\sin x\sin(x+h)} \qquad \left\{\begin{array}{l}\text{Notice how the numerator}\\ \text{was simplified}\end{array}\right\}$$

$$=-\operatorname{cosec}^2 x$$

12. $\boxed{f(x)=e^x}$:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}=\lim_{h\to 0}\frac{e^x\cdot e^h-e^x}{h}$$

$$=e^x\lim_{h\to 0}\frac{e^h-1}{h}$$

$$=e^x$$

13. $\boxed{f(x)=a^x}$:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{a^{x+h}-a^x}{h}$$

$$=a^x\cdot\lim_{h\to 0}\frac{a^h-1}{h}$$

$$=a^x\ln a$$

14. $\boxed{f(x)=\ln x}$:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h}$$

$$=\lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{h}$$

$$=\lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{x\cdot\frac{h}{x}}$$

$$=\frac{1}{x}\qquad\left\{\text{Because}\lim_{\theta\to 0}\left(\frac{\ln(1+\theta)}{\theta}\right)=1\right\}$$

15. $\boxed{f(x)=\log_a x}$:

Since $\log_a x$ can be written as $\dfrac{\ln x}{\ln a}$, $\dfrac{d(\log_a x)}{dx}$ will be $\dfrac{1}{\ln a}\dfrac{d\ln(x)}{dx}$ {because $\dfrac{1}{\ln a}$ is a constant so it can be taken outside the differentiation operator; we will prove the validity of this step later}.

Therefore, $\dfrac{d}{dx}(\log_a x)=\dfrac{1}{x\ln a}$

16. $\boxed{f(x)=\sin^{-1} x}$:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sin^{-1}(x+h)-\sin^{-1}x}{h}$$

> The numerator can be simplified as follows:
>
> Let us consider a general expression $\sin^{-1}x-\sin^{-1}y$.
>
> Let $\sin^{-1}x=p$ and $\sin^{-1}y=q$, so that $x=\sin p$ and $y=\sin q$
>
> Now, $\sin(p-q)=\sin p\cos q-\cos p\sin q$
>
> $$=x\sqrt{1-y^2}-y\sqrt{1-x^2}$$
>
> Therefore,
>
> $p-q=\sin^{-1}x-\sin^{-1}y=\sin^{-1}\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right)$

We use this relation now to simplify the numerator:

$$f'(x)=\lim_{h\to 0}\frac{\sin^{-1}\left\{(x+h)\sqrt{1-x^2}-x\sqrt{1-(x+h)^2}\right\}}{h}$$

$$=\lim_{h\to 0}\frac{\sin^{-1}y}{h}\qquad\left\{\begin{array}{l}\text{we replaced the large}\\ \text{argument of }\sin^{-1}\text{ by }y\end{array}\right\}$$

$$=\lim_{h\to 0}\frac{\sin^{-1}y}{y}\cdot\frac{y}{h}$$

Notice that as $h\to 0,\ y\to 0$ so $\dfrac{\sin^{-1}y}{y}\to 1$

Also,
$$\lim_{h\to 0}\frac{y}{h}=\lim_{h\to 0}\frac{(x+h)\sqrt{1-x^2}-x\sqrt{1-(x+h)^2}}{h}$$

$$=\lim_{h\to 0}\frac{(x+h)^2(1-x^2)-x^2\left(1-(x+h)^2\right)}{h\left\{(x+h)\sqrt{1-x^2}+x\sqrt{1-(x+h)^2}\right\}}$$

$$= \lim_{h \to 0} \frac{h^2 + 2xh}{h\left\{(x+h)\sqrt{1-x^2} + x\sqrt{1-(x+h)^2}\right\}}$$

$$= \frac{2x}{2x\sqrt{1-x^2}}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

Therefore,

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

17. $\boxed{f(x) = \cos^{-1} x}$:

Notice that $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$

Thus, $$\frac{d}{dx}\left(\sin^{-1} x + \cos^{-1} x\right) = \frac{d\left(\pi/2\right)}{dx} = 0$$

$$\Rightarrow \frac{d\left(\cos^{-1} x\right)}{dx} = \frac{-d\left(\sin^{-1} x\right)}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

Note that for the last step, we have used the fact that differentiation operation is distributive over addition, i.e. $(f+g)' = f' + g'$. We will justify this later.

18. $\boxed{f(x) = \tan^{-1} x}$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\tan^{-1}(x+h) - \tan^{-1} x}{h}$$

$$= \lim_{h \to 0} \left\{ \frac{\tan^{-1} \dfrac{h}{1 + x(x+h)}}{h} \right\} \qquad \left(\begin{array}{l} \text{we used } \tan^{-1} A - \tan^{-1} B \\ = \tan^{-1}\left(\dfrac{A-B}{1+AB}\right); \text{ verify this} \end{array} \right)$$

$$= \lim_{h \to 0} \left\{ \frac{\tan^{-1} \dfrac{h}{1 + x(x+h)}}{\dfrac{h}{1 + x(x+h)}} \right\} \cdot \frac{1}{1 + x(x+h)}$$

$$= \frac{1}{1+x^2}$$

19. $\boxed{f(x) = \text{cosec}^{-1}x}$:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\text{cosec}^{-1}(x+h)-\text{cosec}^{-1}x}{h}$$

$$= \lim_{h\to 0}\frac{\sin^{-1}\left(\frac{1}{x+h}\right)-\sin^{-1}\left(\frac{1}{x}\right)}{h} \qquad \left(\begin{array}{l}\text{Notice that}\\ \text{cosec}^{-1}\theta = \sin^{-1}\frac{1}{\theta}\end{array}\right)$$

$$= \lim_{h\to 0}\frac{\sin^{-1}\left\{\left(\frac{1}{x+h}\right)\sqrt{1-\left(\frac{1}{x}\right)^2}-\left(\frac{1}{x}\right)\sqrt{1-\left(\frac{1}{x+h}\right)^2}\right\}}{h}$$

$$= \lim_{h\to 0}\frac{\sin^{-1}\left\{\frac{\sqrt{x^2-1}}{|x|(x+h)}-\frac{\sqrt{(x+h)^2-1}}{x|x+h|}\right\}}{h}$$

$$= \lim_{h\to 0}\frac{\sin^{-1}y}{h} \qquad \left(\begin{array}{l}\text{where } y \text{ is the argument of } \sin^{-1};\\ \text{Note that } y\to 0 \text{ as } h\to 0\end{array}\right)$$

$$= \lim_{h\to 0}\frac{\sin^{-1}y}{y}\cdot\frac{y}{h}$$

Now, it can easily be verified by rationalization that

$$\lim_{h\to 0}\frac{y}{h} = \frac{-1}{|x|\sqrt{x^2-1}}$$

Therefore,

$$f'(x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

20. $\boxed{f(x) = \sec^{-1}x}$:

Notice that $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}$

Therefore, $\frac{d\left(\sec^{-1}x\right)}{dx} = \frac{-d\left(\text{cosec}^{-1}x\right)}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$

21. $\boxed{f(x) = \cot^{-1} x}$:

Notice again that $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

Therefore, as in the earlier cases,

$$\frac{d\left(\cot^{-1} x\right)}{dx} = \frac{-d\left(\tan^{-1} x\right)}{dx} = \frac{-1}{1+x^2}$$

It would be of help to you to get used to these differentiation on formulae as soon as possible, since they will be widely used subsequently.

Section - 3 RULES OF DIFFERENTIATION

In this section, we will see certain general rules pertaining to differentiation that will help us in calculating the derivative of an arbitrary function without using first principles.

In the discussion that follows, we assume that $f(x)$ and $g(x)$ are two differentiable functions

Rule 1: $\boxed{\frac{d\left(kf(x)\right)}{dx} = \frac{kd\left(f(x)\right)}{dx}}$

This rule says that a constant can be taken out from the argument of the differentiation operator. The proof is very straight forward:

$$\frac{d\left(kf(x)\right)}{dx} = \lim_{h \to 0} \frac{kf(x+h) - kf(x)}{h}$$

$$= k \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \frac{kd\left(f(x)\right)}{dx}$$

Rule 2: $\boxed{\frac{d\left(f(x) \pm g(x)\right)}{dx} = \frac{d\left(f(x)\right)}{dx} \pm \frac{d\left(g(x)\right)}{dx}}$

This rule says that the differentiation operator is distributive over addition and subtraction. The proof for this is again quite straightforward:

$$\frac{d\left(f(x)\pm g(x)\right)}{dx}=\lim_{h\to 0}\frac{\{f(x+h)\pm g(x+h)\}-\{f(x)\pm g(x)\}}{h}$$

$$=\lim_{h\to 0}\frac{\{f(x+h)-f(x)\}\pm\{g(x+h)-g(x)\}}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\pm\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=\frac{d\left(f(x)\right)}{dx}\pm\frac{d\left(g(x)\right)}{dx}$$

Rule 3: $$\boxed{\frac{d\{f(x)g(x)\}}{dx}=f(x)\frac{d\left(g(x)\right)}{dx}+g(x)\frac{d\left(f(x)\right)}{dx}}$$

This rule, called the Product rule, is of great help in evaluating the derivative of the product of two (or more) functions.

The proof is as follows:

$$\frac{d\{f(x)g(x)\}}{dx}=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

Introduction of an extra term

$$=\lim_{h\to 0}\frac{f(x+h)g(x+h)\boxed{-f(x+h)g(x)+f(x+h)g(x)}-f(x)g(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)\{g(x+h)-g(x)\}+g(x)\{f(x+h)-f(x)\}}{h}$$

$$=\lim_{h\to 0}f(x+h)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}+g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=f(x)\frac{d\left(g(x)\right)}{dx}+g(x)\frac{d\left(f(x)\right)}{dx}$$

Rule 4: $$\boxed{\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\}=\frac{g(x)\frac{d\left(f(x)\right)}{dx}-f(x)\frac{d\left(g(x)\right)}{dx}}{\left(g(x)\right)^2};\text{ wherever } g(x)\neq 0}$$

This rule, called the Quotient rule, helps us evaluate the derivative of the ratio of two functions $f(x)/g(x)$, wherever $g(x)\neq 0$.

$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\}=\lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h}$$

$$=\lim_{h\to 0}\frac{g(x)f(x+h)-f(x)g(x+h)}{h\cdot g(x)\cdot g(x+h)}$$

Introduction of an extra term

$$=\lim_{h\to 0}\frac{g(x)f(x+h)\boxed{-g(x)f(x)+g(x)f(x)}-f(x)g(x+h)}{h\cdot g(x)\cdot g(x+h)}$$

$$=\lim_{h\to 0}\frac{1}{g(x)\cdot g(x+h)}\lim_{h\to 0}\left[\frac{g(x)\{f(x+h)-f(x)\}}{h}+f(x)\frac{\{g(x)-g(x+h)\}}{h}\right]$$

$$=\frac{1}{(g(x))^2}\cdot\left[g(x)\frac{d(f(x))}{dx}-f(x)\frac{d(g(x))}{dx}\right]$$

$$=\frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$$

Rule 5: $$\boxed{\frac{d\,f(g(x))}{dx}=\frac{d\,f(g(x))}{d(g(x))}\cdot\frac{d(g(x))}{dx}}$$

This can be stated more conveniently as

$$\left(f(g(x))\right)'=f'(g(x))g'(x)$$

This rule, called the Chain rule, is extremely useful to differentiate composite functions, and will be used extensively. It says that to differentiate $f(g(x))$, we first differentiate *f* with respect to $g(x)$ and not *x* (i.e, we treat *g*(*x*) as a variable *y* and differentiate *f* with respect to *y*) and then we multiply this by the derivative of $g(x)$ (or y) with respect to *x*.

$$\frac{d\left[f\left(g\left(x\right)\right)\right]}{dx}=\lim_{h\to 0}\frac{f\left(g\left(x+h\right)\right)-f\left(g\left(x\right)\right)}{h}$$

$$=\lim_{h\to 0}\frac{f\left(g\left(x+h\right)\right)-f\left(g\left(x\right)\right)}{g\left(x+h\right)-g\left(x\right)}\cdot\frac{g\left(x+h\right)-g\left(x\right)}{h}$$

Notice carefully that the first ratio in the limit above is actually the derivative of f, but with respect to $g\left(x\right)$ as the variable $\left\{\text{as } h\to 0, \left(g\left(x+h\right)-g\left(x\right)\right)\to 0\right\}$.

You can view this graphically as follows:

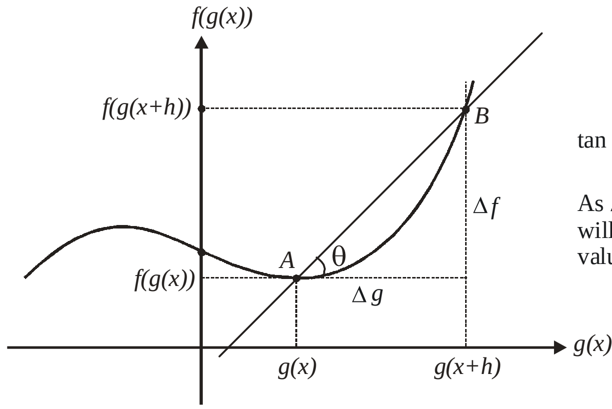

Fig - 4

Therefore,

$$\frac{d\left[f\left(g\left(x\right)\right)\right]}{dx}=\frac{d\ f\left(g\left(x\right)\right)}{d\left(g\left(x\right)\right)}\cdot\frac{d\left(g\left(x\right)\right)}{dx}$$

$$=f'\left(g\left(x\right)\right)\cdot g'\left(x\right)$$

For example,

$$\frac{d\left(\sin\left(x^2\right)\right)}{dx}=\underbrace{\cos\left(x^2\right)}_{\substack{\uparrow\\ \text{Derivative of}\\ \sin(x^2)\ w.r.t\ (x^2)}}\cdot\underbrace{2x}_{\substack{\uparrow\\ \text{Derivative of}\\ (x^2)\ w.r.t\ x}}$$

$$\frac{d\left(\log\left(\sin x\right)\right)}{dx}=\underbrace{\frac{1}{\sin x}}_{\substack{\downarrow\\ \text{Derivative of log}\\ (\sin x)\ w.r.t\ \sin x}}\cdot\underbrace{\cos x}_{\substack{\downarrow\\ \text{Derivative of}\\ \sin x\ w.r.t\ x}}$$

and so on. We will encounter lots examples of the application of this rule soon.

You must note here that for this rule to be applicable at any $x = a$ the function $f(x)$ must be differentiable at $x = g(a)$, since the variable that is the argument (input) to f is $g(x)$ {and not x}.

Example – 1

Evaluate the derivatives of the following functions by first principles:

(a) $f(x) = \log(\sin x)$ **(b)** $f(x) = \sin(\log x)$

(c) $f(x) = \sqrt[3]{\sin x}$ **(d)** $f(x) = e^{x^2}$

Solution: These functions are all compositions of multiple functions and can easily be differentiated using the Chain rule. However, our purpose here is to differentiate them using first principles. This exercise, instead of being considered futile, should be seen as an application of your algebraic manipulation skills.

(a) $$f'(x) = \lim_{h\to 0} \frac{\log(\sin(x+h)) - \log(\sin x)}{h}$$

$$= \lim_{h\to 0} \frac{\log\left(\dfrac{\sin(x+h)}{\sin x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\log\left(\dfrac{\sin x \cos h + \cos x \sin h}{\sin x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\log(\cos h + \cot x \sin h)}{h}$$

$$= \lim_{h\to 0} \frac{\log\{\cos h(1 + \cot x \tan h)\}}{h}$$

$$= \lim_{h\to 0} \frac{\log(\cos h)}{h} + \lim_{h\to 0} \frac{\log(1 + \cot x \tan h)}{h}$$

This limit is 1

$$= \lim_{h\to 0} \frac{\log\{1 + (\cos h - 1)\}}{h} + \lim_{h\to 0} \boxed{\frac{\log(1 + \cot x\ \tan h)}{\cot x\ \tan h}} \cdot \frac{\cot x\ \tan h}{h}$$

This limit is 1

$$= \lim_{h\to 0} \boxed{\frac{\log\{1 + (\cos h - 1)\}}{\cos h - 1}}\ \frac{\cos h - 1}{h} + \cot x$$

$$= \lim_{h\to 0} \frac{\cos h - 1}{h} + \cot x$$

$$= 0 + \cot x$$

$$= \cot x$$

◀

(b) $f'(x) = \lim_{h\to 0} \dfrac{\sin(\log(x+h)) - \sin(\log x)}{h}$

$$= 2\lim_{h\to 0} \frac{\cos\left\{\frac{\log(x+h)+\log(x)}{2}\right\}\sin\left\{\frac{\log(x+h)-\log(x)}{2}\right\}}{h}$$

$$= 2\cos(\log(x))\lim_{h\to 0}\left\{\frac{\sin\left\{\frac{1}{2}\log\left(1+\frac{h}{x}\right)\right\}}{h}\right\}$$

$$= 2\cos(\log(x))\lim_{h\to 0}\left[\left\{\frac{\sin\left\{\frac{1}{2}\log\left(1+\frac{h}{x}\right)\right\}}{\frac{1}{2}\log\left(1+\frac{h}{x}\right)}\right\}\cdot\frac{\frac{1}{2}\log\left(1+\frac{h}{x}\right)}{h}\right]$$

$$= 2\cos(\log(x))\lim_{h\to 0}\left\{\frac{1}{2}\frac{\log\left(1+\frac{h}{x}\right)}{x\cdot h/x}\right\}$$

$$= \frac{\cos(\log(x))}{x}$$ ◀

(c) $f'(x) = \lim_{h\to 0} \dfrac{\sqrt[3]{\sin(x+h)} - \sqrt[3]{\sin x}}{h}$

$$= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h}\cdot\frac{1}{[\sin(x+h)]^{2/3} + (\sin x)^{2/3} + [\sin x\cdot\sin(x+h)]^{1/3}}$$

{This step was accomplished by rationalization}

$$= \frac{\cos x}{3\cdot(\sin x)^{2/3}}$$ ◀

(d) $f'(x)=\lim\limits_{h\to 0}\dfrac{e^{(x+h)^2}-e^{x^2}}{h}$

$$=\lim_{h\to 0}\frac{e^{x^2+h^2+2xh}-e^{x^2}}{h}$$

$$=e^{x^2}\lim_{h\to 0}\frac{e^{h^2+2xh}-1}{h}$$

$$=e^{x^2}\lim_{h\to 0}\left\{\boxed{\frac{e^{h^2+2xh}-1}{h^2+2xh}}\cdot\frac{h^2+2xh}{h}\right\}$$

The limiting value of this ratio will be 1

$$=e^{x^2}\cdot 2x$$ ◀

Example – 2

Evaluate the derivatives of the following functions using the techniques developed in section - 3:

(a) $f(x)=(x^2+3x)\sin^{-1}x$ **(b)** $f(x)=\tan\left(\dfrac{x+1}{x+2}\right)$

(c) $f(x)=\cos^{-1}\left(\sqrt{1-x^2}\right)+\dfrac{x^2+2x+3}{x+2}$ **(d)** $f(x)=\sqrt{4-\tan^{-1}x^2}$

Solution: **(a)** This function can be differentiated by the Product Rule (Rule - 3)

$$\frac{d(f(x))}{dx}=(x^2+3x)\frac{d(\sin^{-1}x)}{dx}+\sin^{-1}x\frac{d(x^2+3x)}{dx}$$

$$=\frac{(x^2+3x)}{\sqrt{1-x^2}}+(\sin^{-1}x)\cdot(2x+3)$$ ◀

(b) This function will be differentiated by the Chain rule (Rule 5); the argument $\left(\dfrac{x+1}{x+2}\right)$ can be differentiated by the Quotient rule (Rule 4):

$$\frac{d(f(x))}{dx}$$

$$=\frac{d\left\{\tan\left(\frac{x+1}{x+2}\right)\right\}}{d\left(\frac{x+1}{x+2}\right)}\cdot\frac{d\left(\frac{x+1}{x+2}\right)}{dx}$$

$$= \sec^2\left(\frac{x+1}{x+2}\right)\cdot\frac{(x+2)\cdot(1)-(x+1)\cdot(1)}{(x+2)^2}$$

$$\frac{1}{(x+2)^2}\cdot\sec^2\left(\frac{x+1}{x+2}\right)$$ ◀

(c) Here, we will have to use a combination of these techniques as follows:

$$\frac{d\left(f(x)\right)}{dx} = \left\{\frac{-1}{1-\left(\sqrt{1-x^2}\right)^2}\right\}\cdot\frac{-x}{\sqrt{1-x^2}} + \frac{(x+2)(2x+2)-\left(x^2+2x+3\right)\cdot(1)}{(x+2)^2}$$

$$= \frac{1}{x\sqrt{1-x^2}} + \frac{x^2+4x+1}{(x+2)^2}$$ ◀

You are urged to work out the solution on your own.

(d) $$\frac{d\left(f(x)\right)}{dx} = \frac{1}{2\sqrt{4-\tan^{-1}x^2}}\cdot\frac{-1}{1+x^4}\cdot 2x$$

$$= \frac{-x}{\left(1+x^4\right)\sqrt{4-\tan^{-1}x^2}}$$ ◀

Example – 3

If $f'(x) = g(x)$, find the derivative of $f^{-1}(x)$

Solution: To evaluate the required derivative, we can apply the chain rule on the relation:

$$f\left(f^{-1}(x)\right) = x$$

$$\Rightarrow \frac{d}{dx}\left\{f\left(f^{-1}(x)\right)\right\} = \frac{d}{dx}(x) = 1$$

$$\Rightarrow f'\left(f^{-1}(x)\right)\frac{d}{dx}\left(f^{-1}(x)\right) = 1$$

$$\Rightarrow \frac{d}{dx}\left(f^{-1}(x)\right)\frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{g\left(f^{-1}(x)\right)}$$

For example, we know that $\frac{d}{dx}(\sin x) = \cos x$

$$\Rightarrow \frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\cos\left(\sin^{-1}(x)\right)}$$

$$= \frac{1}{\cos\left(\cos^{-1}\sqrt{1-x^2}\right)}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

In this way, we can evaluate the derivative of any inverse function, given the derivative of the original function. ◀

Section - 4 DIFFERENTIATION OF PARAMETRIC/IMPLICIT FUNCTIONS

(A) PARAMETRIC FUNCTIONS

Sometimes, when expressing y as a function of x, one might not use a direction relation between x and y; instead, one might express both x and y as functions of a third variable, say t:

$$x = f(t)$$

$$y = g(t)$$

In that case, how would $\frac{dy}{dx}$ be evaluated?

One option is to eliminate the parameter t and obtain a relation involving only x and y, from which $\frac{dy}{dx}$ may be obtained; however, this could lead to cumbersome expressions.

Another alternative can be taken as follows; we rearrange $\frac{dy}{dx}$ to involve t also:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

This relation says that for evaluating the derivative of y w.r.t x, we evaluate the derivative of y and x w.r.t the parameter t, and then take their ratio.

Let us try this on some examples:

(i) $x = r\cos\theta$ $\quad y = r\sin\theta$; r is a constant

This parametric relation represents a circle of radius r. We will follow both the approaches to determine $\frac{dy}{dx}$:

⇒ **ELIMINATION:**

Square and add the two relations for x and y to obtain:

$$x^2 + y^2 = r^2$$

$$\Rightarrow y = \pm\sqrt{r^2 - x^2}$$

$$\Rightarrow \frac{dy}{dx} = \pm\frac{x}{\sqrt{r^2 - x^2}} \quad \left(\begin{array}{l}\text{For each } x, \text{ we obtain two } y' \\ \text{values because the curve is a circle}\end{array}\right)$$

⇒ **PARAMETRIC DIFFERENTIATION**

$$\frac{dx}{d\theta} = -r\sin\theta \qquad \frac{dy}{d\theta} = r\cos\theta$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\cot\theta$$

(ii) $x = a\cos\theta$ $\quad y = b\sin\theta$; a, b are constants

This parametric relation represents an ellipse with major and minor axis 2a and 2b respectively.

⇒ **ELIMINATION**

θ can easily be eliminated to obtain:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

$$\Rightarrow \frac{dy}{dx} = \pm\frac{b}{a}\frac{x}{\sqrt{a^2 - x^2}}$$

⇒ **PARAMETRIC DIFFERENTIATION**

$$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-b}{a}\cot\theta$$

In later examples, we will observe that in many cases, parametric differentiation turns out to be much more convenient than differentiation after elimination.

(B) IMPLICIT FUNCTIONS

Sometimes, the relation between the variables x and y is specified in the form $f(x, y) = 0$ that is, y is not explicitly specified in terms of x, since this explicit expression is either not possible or not convenient. In such a case, y is said to be an implicit function of x.

How do we find $\frac{dy}{dx}$ in such a case?

We simply differentiate the relation $f(x, y) = 0$ with respect to x, using $\frac{dy}{dx}$ for the derivative of the variable y. Then we solve for $\frac{dy}{dx}$.

This will become clear from some examples:

$$\Rightarrow \quad x^2 + y^2 = 1$$

Differentiating both sides w.r.t x:

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

$$\Rightarrow x^3 + y^3 + 2xy = 2$$

Differentiating both sides w.r.t x:

$$3x^2 + 3y^2\frac{dy}{dx} + 2x\frac{dy}{dx} + 2y = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\left(3x^2 + 2y\right)}{3y^2 + 2x}$$

$$\Rightarrow y = \cos(x + y)$$

Differentiating both sides w.r.t x:

$$\frac{dy}{dx} = -\sin(x + y)\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin(x + y)}{1 + \sin(x + y)}$$

Observe that in case of differentiation of implicit functions, the expression for the derivative $\frac{dy}{dx}$ will generally not be independent of y.

Example – 4

If $x^2 + y^2 = t + \frac{1}{t}$ and $x^4 + y^4 = t^2 + \frac{1}{t^2}$, then prove that $\frac{dy}{dx} = \frac{-1}{x^3 y}$

Solution: We first try to use the two given relations to get rid of the parameter t, so that we obtain a (implicit) relation between x and y.

$$x^2 + y^2 = t + \frac{1}{t}$$

Squaring, we get

$$x^4 + y^4 + 2x^2y^2 = t^2 + \frac{1}{t^2} + 2 \qquad \text{... (i)}$$

Using the second relation in (i), we get

$$2x^2y^2 = 2$$

$$\Rightarrow y^2 = \frac{1}{x^2}$$

Differentiating both sides w.r.t x, we get

$$2y\frac{dy}{dx} = \frac{-2}{x^3}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{x^3 y}$$

◀

Example – 5

If $y^2 = a^2\cos^2 x + b^2 \sin^2 x,$ then prove that

$$\frac{d^2y}{dx^2} + y = \frac{a^2b^2}{y^3}$$

Solution: The final relation that we need to obtain is independent of $\sin x$ and $\cos x$; this gives us a hint that using the given relation, we must first get rid of $\sin x$ and $\cos x$:

$$y^2 = a^2\cos^2 x + b^2\sin^2 x$$

$$= \frac{1}{2}\left\{a^2\left(2\cos^2 x\right) + b^2\left(2\sin^2 x\right)\right\}$$

$$= \frac{1}{2}\left\{a^2\left(1+\cos 2x\right) + b^2\left(1-\cos 2x\right)\right\}$$

$$= \frac{1}{2}\left\{\left(a^2+b^2\right) + \left(a^2-b^2\right)\cos 2x\right\}$$

$$\Rightarrow 2y^2 - \left(a^2+b^2\right) = \left(a^2-b^2\right)\cos 2x \qquad \text{... (i)}$$

Differentiating both sides of (i) w.r.t x, we get

$$4y\frac{dy}{dx} = -2\left(a^2 - b^2\right)\sin 2x$$

$$\Rightarrow -2y\frac{dy}{dx} = \left(a^2 - b^2\right)\sin 2x \qquad \text{... (ii)}$$

We see now that squaring (i) and (ii) and adding them will lead to an expression independent of the trig. terms:

$$\left(2y^2 - \left(a^2 + b^2\right)\right)^2 + 4y^2\left(\frac{dy}{dx}\right)^2 = \left(a^2 - b^2\right)^2$$

A slight rearrangement gives:

$$\left(\frac{dy}{dx}\right)^2 + y^2 - \left(a^2 + b^2\right) = -\frac{a^2b^2}{y^2} \qquad \text{... (iii)}$$

Differentiating both sides of (iii) w.r.t x:

$$2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + 2y\frac{dy}{dx} = \frac{2a^2b^2}{y^3}\frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} + y = \frac{a^2b^2}{y^3}$$

◀

Example – 6

If the derivatives of $f(x)$ and $g(x)$ are known, find the derivative of $y = \{f(x)\}^{g(x)}$

Solution: We cannot directly differentiate the given relation since no rule tells us how to differentiate a term p^q where both p and q are variables.

What we can instead do is take the logarithm of both sides of the given relation:

$$y = \{f(x)\}^{g(x)}$$

$$\Rightarrow \ln y = g(x)\ln\left(f(x)\right)$$

Now we differentiate both sides w.r.t x:

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = g(x)\cdot\frac{1}{f(x)}\cdot f'(x) + \ln\left(f(x)\right)\cdot g'(x)$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{f'(x)g(x)}{f(x)} + g'(x)\ln\left(f(x)\right)\right.$$

$$= \left(f(x)\right)^{g(x)}\left\{\frac{f'(x)g(x)}{f(x)} + g'(x)\ln\left(f(x)\right)\right\}$$

As a simple example, suppose we have to differentiate $y = x^x$:

$$\ln y = x \ln x$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \ln x \cdot 1$$

$$= 1 + \ln x$$

$$\Rightarrow \frac{dy}{dx} = y(1 + \ln x)$$

$$= x^x (1 + \ln x)$$

◀

Section - 5 | L'HOSPITAL' RULE

We had made a mention of the L'Hospital's Rule (abbreviated as the LH rule) in the unit on limits. We had deferred the introduction of this rule to this chapter since it requires the use of differentiation. The LH rule can be used to evaluate limits that are of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Consider two functions $f(x)$ and $g(x)$ which are differentiable in the neighbourhood of the point $x = a$ (except possibly at the point $x = a$ itself). Let $g'(x) \neq 0$ in this neighbourhood.

The LH rule says that

if $\quad \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$

or if $\quad \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty$

then $\quad \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

provided that the limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ exists. Although the LH rule is applicable to limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, you should be able to understand that other indeterminate forms like $0.\infty,\ \infty - \infty,\ 1^{\infty},\ \infty^0$ or 0^0 can be reduced to these two indeterminate forms using appropriate algebtain manipulations.

You are urged to think of some (non-rigorous) justifiction for this rule.

Lets apply this rule on some examples.

Example – 7

Evaluate:

(a) $\lim\limits_{x\to 0} x\ln x$ **(b)** $\lim\limits_{x\to\infty} \dfrac{\ln x}{x}$

Solution: We have encountered both these limits in the unit on Limits. Here, we'll re-evaluate them using the LH rule.

(a)

$$L = \lim_{x\to 0} x\ln x \qquad (0\times -\infty \text{ form})$$

$$= \lim_{x\to 0} \frac{\ln x}{1/x} \qquad \left(\frac{-\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{1/x}{-1/x^2} \qquad \text{(By applying the LH rule)}$$

$$= \lim_{x\to 0}(-x)$$

$$= 0$$

This is what we got earlier

(b)

$$L = \lim_{x\to\infty} \frac{\ln x}{x} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to\infty} \frac{1/x}{1} \qquad \text{(By applying the LH rule)}$$

$$= 0$$

Example – 8

Evaluate $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x}\right)^{\sin x}$

Solution: This limit is of the indeterminate form ∞^0. Lets first convert it into the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

$$L = \lim_{x\to 0^+} \left(\frac{1}{x}\right)^{\sin x}$$

$$= \lim_{x\to 0^+} \sin x \cdot \ln\left(\frac{1}{x}\right) \qquad \left\{\begin{matrix}\text{Doing this is justified since} \\ x>0 \ln\left(\dfrac{1}{x}\right) \text{is defined}\end{matrix}\right\}$$

$$= -\lim_{x\to 0^+} \sin x \cdot \ln x$$

$$= -\lim_{x\to 0^+} \frac{\ln x}{\operatorname{cosec} x} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= -\lim_{x\to 0^+} \frac{1/x}{-\operatorname{cosec} \cot x} \qquad \text{(By applying the LH rule)}$$

$$= \lim_{x\to 0^+} \frac{\sin^2 x}{x \cos x}$$

$$= \lim_{x\to 0^+} \left(\frac{\sin x}{x}\right) \cdot \tan x$$

$$= 0$$

Example – 9

Evaluate $\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{x}{\ln x}\right)$

Solution:

$$L = \lim_{x\to 1}\left(\frac{1-x}{\ln x}\right) \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 1}\left(\frac{-1}{1/x}\right) \qquad \left(\text{By applying the LH rule}\right)$$

$$= \lim_{x\to 1}(-x)$$

$$= -1$$

Example – 10

Evaluate $\lim_{x\to\infty}\left(\frac{x^2 + \sin x}{x^2}\right)$

Solution: The limit is of the indeterminate form $\frac{\infty}{\infty}$, so we apply the L.H. rule:

$$L = \lim_{x\to\infty}\left(\frac{x^2 + \sin x}{x^2}\right) \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to\infty}\left(\frac{2x + \cos x}{2x}\right) \qquad \left(\frac{\infty}{\infty} \text{ form agin}\right) \qquad ...(1)$$

$$= \lim_{x\to\infty}\left(\frac{2 - \sin x}{2}\right) \qquad ...(2)$$

$$= 1 - \frac{1}{2}\lim_{x\to\infty}(\sin x)$$

Now, we know that $\lim_{x\to\infty}(\sin x)$ does not exist since sin x is an oscillating function and does not converge to any particular value. What does this imply for our current limit? Does it not exist ? Think about the expression $\left(\frac{x^2 + \sin x}{x^2}\right)$ carefully:

$$\lim_{x\to\infty}\left(\frac{x^2 + \sin x}{x^2}\right) = \lim_{x\to\infty}\left(1 + \frac{\sin x}{x^2}\right)$$

$$= 1 + \lim_{x\to\infty}\left(\frac{\sin x}{x^2}\right)$$

$$= 1 + 0 \qquad (\because \sin x \text{ is bounded})$$

$$= 1$$

Thus, a limit does infact exists while the LH rule says that it does not exist. Why?

This is because the LH rule is not applicable here. Go back to the definition of the LH rule which says that $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$ if the latter limit exists.

In this example, you cannot apply the LH rule on the expression in (1) since the limit for the expression obtained after differentiation (the one in (2)) does not exist.

Thus, the LH rule must be used with care.