Indefinite Integration

CONCEPT NOTES

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Indefinite Integration

As mentioned in the unit "**Integration Basics**", this chapter is devoted entirely to developing tools and techniques to find out anti-derivatives of arbitrary functions. For readers who have not read "**Integration Basics**", it is advisable to go through that chapter first, before reading this.

Section - 1 BASIC RULES AND FORMULAE

The following is a set of straight forward rules pertaining to integration, that follow by definition:

(a) A constant is always included in the expression for the indefinite integral, i.e.,

if $g'(x) = f(x)$, then

$$\int f(x)dx = g(x) + C$$

This is because, as mentioned earlier, the derivative of a constant is 0.

(b) The integral of a derivative gives back the same function itself (with a constant):

$$\int f'(x)dx = f(x) + C$$

The derivative of an integral also gives the same function:

$$\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$$

These two results are in agreement with the fact that differentiation and integration are inverse operations.

(c) $\int \{f(x) \pm g(x)\}dx = \int f(x)dx + \int g(x)dx$

(d) $\int k\, f(x)dx = k\int f(x)dx$

(e) If $\int f(x)dx = g(x) + C$, then

$$\int f(ax+b)dx = \frac{1}{a}g(ax+b) + C \qquad \text{... (i)}$$

How is this true? Since $g(x)$ is the anti-derivative of $f(x)$, $g'(x) = f(x)$.

Now we differentiate (i) :

$$f(ax+b) = \frac{1}{a}\frac{d}{dx}\left(g(ax+b)\right)$$

$$= \frac{1}{a}\cdot g'(ax+b)\cdot a$$

$$= g'(ax+b)$$

$$= f(ax+b)$$

This shows that (i) holds true.

This result is quite useful as we'll realise in the course of studying this chapter.

We now present a table of some basic integration formulae. You are urged to verify the truth of these formulae by differentiating the right hand side of each formula and check whether the expression you obtain is equal to the one inside the integral on the left hand side, or not:

BASIC INTEGRATION FORMULAE

01.	$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad ; \quad n \neq -1$ *	**11.**	$\int \cot x dx = \ln\lvert\sin x\rvert + C$
02.	$\int \frac{1}{x} dx = \ln x + C$	**12.**	$\int \tan x dx = -\ln\lvert\cos x\rvert + C$
03.	$\int e^x dx = e^x + C$	**13.**	$\int \sec x dx = \ln\lvert\sec x + \tan x\rvert + C$
04.	$\int a^x dx = \frac{a^x}{\ln a} + C$	**14.**	$\int \operatorname{cosec} x dx = \ln\lvert\operatorname{cosec} x - \cot x\rvert + C$
05.	$\int \sin x dx = -\cos x + C$	**15.**	$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a} + C$
06.	$\int \cos x dx = \sin x + C$	**16.**	$\int \frac{-1}{\sqrt{a^2 - x^2}} dx = -\sin^{-1}\frac{x}{a} + C_1$ $= \cos^{-1}\frac{x}{a} + C_2$
07.	$\int \sec^2 x dx = \tan x + C$	**17.**	$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$
08.	$\int \operatorname{cosec}^2 x dx = -\cot x + C$	**18.**	$\int \frac{-1}{a^2 + x^2} dx = \frac{-1}{a}\tan^{-1}\frac{x}{a} + C_1$ $= \frac{1}{a}\cot^{-1}\frac{x}{a} + C_2$
09.	$\int \sec x \tan x dx = \sec x + C$	**19.**	$\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$
10.	$\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + C$	**20.**	$\int \frac{-1}{x\sqrt{x^2 - a^2}} dx = \frac{-1}{a}\sec^{-1}\frac{x}{a} + C_1$ $= \frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a} + C_2$
	* In particular, $\int dx = x + C$		

For example, to prove that formula (17) is true, we differentiate the right side:

$$\frac{d}{dx}\left(\frac{1}{a}\tan^{-1}\frac{x}{a}\right)=\frac{1}{a}\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right)$$

$$=\frac{1}{a}\frac{1}{1+\frac{x^2}{a^2}}\cdot\frac{1}{a}$$

$$=\frac{1}{a^2}\cdot\frac{a^2}{x^2+a^2}$$

$$=\frac{1}{x^2+a^2}$$

Therefore, $$\int\frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}\frac{x}{a}+C$$

Suppose that you now have to evaluate the following integral:

$$\int\frac{1}{a^2+(bx+c)^2}dx$$

We will use formula (17) and rule (*e*) stated earlier:

$$\int\frac{1}{a^2+(bx+c)^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{bx+c}{a}\right)\times\frac{1}{b}+C$$

$$=\frac{1}{ab}\tan^{-1}\left(\frac{bx+c}{a}\right)+C$$

Observe carefully how we obtained the final expression.

There is a variety of methods in which we can evaluate indefinite integrals. We can broadly divide these methods into five major categories:

(1) SIMPLE REARRANGEMENTS : We rearrange the given expression in such a way so that we obtain a combination of the basic integrals that we have just discussed.

(2) SUBSTITUTIONS : We use some substitution to convert the given expression into a more conveniently "integrable" form.

(3) EXPANSION USING PARTIAL FRACTIONS : This method is applicable to rational algebraic functions; we use a partial fractions expansion to split such a function into more elementary functions that can easily be integrated.

(4) INTEGRATION BY PARTS : This powerful method can be applied to the product of *any* two arbitrary functions.

(5) REDUCTION FORMULAE : These formulae make it possible to reduce an integral depending on the index $n>0$, called the order of the integral, to an integral of the same type but with a smaller index.

All these methods will now be discussed in detail.

A word of advice: make it a point to practice as much questions as possible for integration; only then can you get the 'hang' of it. You should even attempt the solved examples on your own before looking at the solutions.

Section - 2 INTEGRATION BY SIMPLE REARRANGEMENTS

A lot many functions that we'll encounter can be reduced to simpler forms by some rearrangement/algebraic manipulation. These simpler forms are easily integrable because they correspond to one of the standard basic integrals that we discussed in the previous section.

The rearrangement technique is best illustrated through examples.

Example – 1

Evaluate $\int \frac{\cos x - \cos 2x}{1 - \cos x} dx$.

Solution: We can try expanding $\cos 2x$ by the half angle formula:

$$\int \frac{\cos x - \cos 2x}{1 - \cos x} dx = \int \frac{\cos x - (2\cos^2 x - 1)}{1 - \cos x} dx$$

$$= \int \frac{(2\cos x + 1)(1 - \cos x)}{(1 - \cos x)} dx$$

$$= \int (2\cos x + 1) dx$$

$$= 2\sin x + x + C.$$

Observe how the rearrangement we used led to a simpler expression that was easily integrable.

Example – 2

Evaluate $\int \frac{1}{\sin(x-a)\cos(x-b)} dx$.

Solution: The denominator is of the form $\sin P \cos Q$, where P and Q are variable; but notice an important fact: $P - Q$ is a constant. This should give us the required hint:

$$\int \frac{1}{\sin(x-a)\cos(x-b)} dx = \frac{1}{\cos(a-b)} \int \frac{\cos(a-b)}{\sin(x-a)\cos(x-b)} dx$$

$$= \frac{1}{\cos(a-b)} \int \frac{\cos\{(x-b)-(x-a)\}}{\sin(x-a)\cos(x-b)} dx$$

$$= \frac{1}{\cos(a-b)} \int \frac{\cos(x-b)\cos(x-a) + \sin(x-b)\sin(x-a)}{\sin(x-a)\cos(x-b)} dx$$

$$= \frac{1}{\cos(a-b)} \int \{\cot(x-a) + \tan(x-b)\} dx$$

$$= \frac{1}{\cos(a-b)} \{\ln|\sin(x-a)| - \ln|\cos(x-b)|\} + C$$

$$= \frac{1}{\cos(a-b)} \ln\left|\frac{\sin(x-a)}{\cos(x-b)}\right| + C$$ ◀

What we had to do in this question was therefore to realise that since $P-Q$ is a constant, an introduction of the term cos $(P-Q)$ in the numerator would lead to cancellations and simple 'cot' and 'tan' terms which can easily be integrated.

Example – 3

Evaluate $\int \frac{x^3}{(x+1)^2} dx$.

Solution: The numerator has a degree higher then the denominator which hints that some reduction of this rational expression is possible. This reduction can be accomplished if we somehow rearrange the numerator in such a way that it leads to a cancellation of common factors with the denominator; since the denominator is $(x+1)^2$, we try to rearrange the numerator in terms of $(x+1)$:

$$\int \frac{x^3}{(x+1)^2} dx = \int \frac{(x^3+1)-1}{(x+1)^2} dx$$

$$= \int \left\{ \frac{(x+1)(x^2-x+1)}{(x+1)^2} - \frac{1}{(x+1)^2} \right\} dx$$

$$= \int \left\{ \frac{x^2-x+1}{(x+1)} - \frac{1}{(x+1)^2} \right\} dx$$

$$= \int \left\{ \frac{x^2-x-2+3}{x+1} - \frac{1}{(x+1)^2} \right\} dx$$

$$= \int \left\{ \frac{(x+1)(x-2)+3}{(x+1)} - \frac{1}{(x+1)^2} \right\} dx$$

$$= \int \left\{ (x-2) + \frac{3}{x+1} - \frac{1}{(x+1)^2} \right\} dx$$

$$= \frac{x^2}{2} - 2x + 3\ln(x+1) + \frac{1}{x+1} + C$$ ◀

Example – 4

Evaluate $\int \frac{x}{\sqrt{x+a}+\sqrt{x+b}}\,dx$.

Solution: The form of the expression in the denominator clearly hints that a reduction is possible by rationalization which would lead to a constant term in the denominator:

$$\int \frac{x}{\sqrt{x+a}+\sqrt{x+b}}\,dx = \int \frac{x\left\{\sqrt{x+a}-\sqrt{x+b}\right\}}{(x+a)-(x+b)}dx$$

$$= \frac{1}{(a-b)}\int\left\{x\sqrt{x+a}-x\sqrt{x+b}\right\}dx$$

$$= \frac{1}{a-b}\int\left\{(x+a-a)\sqrt{x+a}-(x+b-b)\sqrt{x+b}\right\}dx$$

$$= \frac{1}{a-b}\int\left\{(x+a)^{3/2}-(x+b)^{3/2}-a(x+a)^{1/2}+b(x+b)^{1/2}\right\}dx$$

$$= \frac{1}{a-b}\left\{\frac{(x+a)^{5/2}}{5/2}-\frac{(x+b)^{5/2}}{5/2}-\frac{a(x+a)^{3/2}}{3/2}+\frac{b(x+b)^{3/2}}{3/2}\right\}+C \quad ◀$$

The first simplification by rationalization led to an expression which involved two terms of the form $x\sqrt{x+a}$; to integrate these terms, we wrote the x outside the root as $(x + a - a)$ so that a final expression is obtained which contains only terms of the form $(x+k)^n$; these could then be integrated easily.

Example – 5

Evaluate $\int \frac{1+\cos 4x}{\cot x-\tan x}\,dx$.

Solution: We simply both the numerator and the denominator separately :

$$\int \frac{1+\cos 4x}{\cot x-\tan x}\,dx = \int \frac{2\cos^2 2x}{\frac{\cos x}{\sin x}-\frac{\sin x}{\cos x}}\,dx$$

$$= \int \frac{2\sin x\cos x\cos^2 2x}{\cos^2 x-\sin^2 x}\,dx$$

$$= \int \frac{\sin 2x\cdot\cos^2 2x}{\cos 2x}\,dx$$

$$= \int \sin 2x\cos 2x \quad dx$$

$$= \frac{1}{2}\int \sin 4x \; dx$$

$$= -\frac{1}{8}\cos 4x + C \quad ◀$$

Example – 6

Evaluate $\int \tan x \tan 2x \tan 3x \, dx$.

Solution: Notice that $3x = 2x + x,$ so that

$$\tan 3x = \tan(2x + x)$$

$$= \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$$

$$\Rightarrow \tan x \tan 2x \tan 3x = \tan 3x - \tan 2x - \tan x$$

The required integral is now easy to evaluate :

$$\int \tan x \tan 2x \tan 3x \; dx = \int \{\tan 3x - \tan 2x - \tan x\} \; dx$$

$$= \frac{1}{3}\ln|\sec 3x| - \frac{1}{2}\ln|\sec 2x| - \ln|\sec x| + C$$ ◀

Example – 7

Evaluate $\int \frac{\sin(x-a)}{\sin(x-b)} dx$.

Solution: Taking cue from Example-2, our aim should be to somehow get rid of the variable term $\sin(x-b)$ in the denominator; to do this, we write the numerator $\sin(x-a)$ as $\sin\{(x-b)-(a-b)\}$:

$$\int \frac{\sin(x-a)}{\sin(x-b)} dx = \int \frac{\sin\{(x-b)-(a-b)\}}{\sin(x-b)} dx$$

$$= \int \frac{\sin(x-b)\cos(a-b) - \cos(x-b)\sin(a-b)}{\sin(x-b)} dx$$

$$= \int \{\cos(a-b) - \sin(a-b)\cot(x-b)\} dx$$

$$= x\cos(a-b) - \sin(a-b)\ln|\sin(x-b)| + C$$ ◀

Example – 8

Evaluate $\int \frac{ax+b}{cx+d} dx$.

Solution: As in Ex-2 and Ex-7, we try to express the numerator in terms of the denominator, so that a cancellation is possible and we can get rid of the variable term $(cx+d)$ in the denominator:

$$ax+b=a\left(x+\frac{b}{a}\right)$$

$$=\frac{a}{c}\left(cx+\frac{bc}{a}\right)$$

$$=\frac{a}{c}\left(cx+d+\frac{bc}{a}-d\right)$$

$$=\frac{a}{c}(cx+d)+\left(b-\frac{ad}{c}\right)$$

$$\Rightarrow \int\frac{ax+b}{cx+d}dx=\int\frac{\frac{a}{c}(cx+d)+\left(b-\frac{ad}{c}\right)}{cx+d}dx$$

$$=\int\left\{\frac{a}{c}+\frac{\left(b-\frac{ad}{c}\right)}{cx+d}\right\}dx$$

$$=\frac{ax}{c}+\frac{1}{c}\left(b-\frac{ad}{c}\right)\ln(cx+d)+C$$

$$=\frac{ax}{c}+\left(\frac{bc-ad}{c^2}\right)\ln(cx+d)+C$$ ◀

Example – 9

Evaluate $\int\frac{\sin 4x}{\sin x}dx$.

Solution: The expression can be simplified by a straight forward expansion of the numerator :

$$\int\frac{\sin 4x}{\sin x}dx=\int\frac{2\sin 2x\cos 2x}{\sin x}dx$$

$$=4\int\frac{\sin x\cos x\cos 2x}{\sin x}dx$$

$$=4\int\cos x\cos 2x\ dx$$

$$=2\int\{\cos 3x+\cos x\}dx$$

$$=\frac{2}{3}\sin 3x+2\sin x+C$$ ◀

Example – 10

Evaluate the following integrals:

(a) $\int \cos^3 x \, dx$ **(b)** $\int \cos^4 x \, dx$

(c) $\int \sin 2x \cos 4x \cos 5x \, dx$ **(d)** $\int \sin^3 x \cos^3 x \, dx$

Solution: **(a)** We know the integral of cos x; we must express the cubic cos term ($\cos^3 x$) in terms of linear cos terms; this can be done using the triple angle formula :

$$\cos 3x = 4\cos^3 x - 3\cos x$$

$$\Rightarrow \cos^3 x = \frac{1}{4}\{3\cos x + \cos 3x\}$$

$$\Rightarrow \int \cos^3 x \, dx = \frac{3}{4}\sin x + \frac{1}{12}\sin 3x + C$$ ◀

(b) Here again, we need to express the fourth degree cos term $(\cos^4 x)$ in terms of linear cos terms:

$$\cos^4 x = (\cos^2 x)^2$$

$$= \left(\frac{1+\cos 2x}{2}\right)^2$$

$$= \frac{1}{4} + \frac{\cos^2 2x}{4} + \frac{1}{2}\cos 2x$$

$$= \frac{1}{4} + \frac{(1+\cos 4x)}{8} + \frac{1}{2}\cos 2x$$

$$= \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$$

$$\Rightarrow \int \cos^4 x \, dx = \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$$ ◀

(c) $$\sin 2x \cos 4x \cos 5x = \frac{1}{2}(2\sin 2x \cos 4x)\cos 5x$$

$$= \frac{1}{2}(\sin 6x - \sin 2x)\cos 5x$$

$$= \frac{1}{4}\{(2\sin 6x \cos 5x) - (2\sin 2x \cos 5x)\}$$

$$= \frac{1}{4}\{\sin 11x + \sin x - \sin 7x + \sin 3x\}$$

$$= \frac{1}{4}\{\sin x + \sin 3x - \sin 7x + \sin 11x\}$$

$$\Rightarrow \int \sin 2x \cos 4x \cos 5x \; dx = \frac{-\cos x}{4} - \frac{\cos 3x}{12} + \frac{\cos 7x}{28} - \frac{\cos 11x}{44} + C$$ ◀

(d) $$\sin^3 x \cos^3 x = (\sin x \cos x)^3$$

$$= \frac{(2\sin x \cos x)^3}{8}$$

$$= \frac{\sin^3 2x}{8}$$

$$= \frac{1}{8}\left\{\frac{3\sin 2x - \sin 6x}{4}\right\} \qquad \left(\text{Triple angle formula}\right)$$

$$= \frac{3}{32}\sin 2x - \frac{1}{32}\sin 6x$$

$$\Rightarrow \int \sin^3 x \cos^3 x \; dx = \frac{-3}{64}\cos 2x + \frac{1}{192}\cos 6x + C$$ ◀

Section - 3	INTEGRATION BY SUBSTITUTION

A lot many times, we will encounter functions whose integrals cannot be obtained from their original expressions; however, an appropriate substitution might reduce the given function to another function whose integral is obtainable.

This method of integration by substitution is used extensively to evaluate integrals. As we progress along this section we will develop certain rules of thumb that will tell us what substitutions to use where. Also, multiple substitutions might be possible for the same function. Therefore, integration by substitution is more of an art and you can develop the knack of it only by extensive practice (and of course, some thinking !)

Example – 11

Evaluate $\int \frac{\cos^3 x}{\sin^2 x + \sin x} dx$

Solution: The general approach while substitution is as follows:

Suppose we have to obtain $\int f(x)dx$. We find some function $\phi(x)$ (and put it equal to a variable *y*) such that $\phi'(x)dx = dy$ is some part of *f*(*x*)*dx*. This will let us express $\int f(x)dx$ in terms of another integral which contains only *y*.

This approach will become quite clear when we apply it on the given example:

$$I = \int \frac{\cos^3 x}{\sin^2 x + \sin x} dx = \int \frac{\left(1 - \sin^2 x\right)\cos x}{\sin^2 x + \sin x} dx$$

Observe carefully why we wrote $\cos^2 x$ in the numerator of *I* as $\left(1 - \sin^2 x\right)$; if we now substitute $\sin x = y$, we'll get $\cos x\, dx = dy$, so that the entire expression of *I* can be reduced to another integral which contains only *y*:

$$\sin x = y$$

$$\cos x\, dx = dy$$

$$I = \int \frac{\left(1 - y^2\right)}{y^2 + y} dy$$

$$= \int \frac{(1-y)}{y} dy$$

$$= \int \left(\frac{1}{y} - 1\right) dy$$

$$= \ln|y| - y + C$$

We see that the modified integral (the integral in terms of *y*) was easily integrable; to obtain the integral in terms of *x*, we now simply substitution $y = \sin x$:

$$I = \ln|\sin x| - \sin x + C$$ ◀

Example – 12

Evaluate $\int \frac{x^7}{\left(1 - x^2\right)^5} dx$

Solution: This example will serve to show that multiple substitutions are possible for the same function.

(a) Notice that the numerator, $x^7 dx$, can be written as $x^6 \cdot x dx$. If we substitute $x^2 = y$, we'll obtain $xdx = \frac{dy}{2}$ so that the entire integral can be expressed in terms of *y*. However, the integral will become

$$\frac{1}{2} \int \frac{y^3}{(1-y)^5} dy$$

which still cannot be integrated directly because of the denominator $(1-y)^5$. What we therefore do is substitute $(1-x^2)=y$ instead of $x^2=y$, because then the denominator will be reduced further directly:

$$1-x^2=y$$

$$\Rightarrow xdx=\frac{-dy}{2}$$

$$I=\int\frac{x^7}{\left(1-x^2\right)^5}dx=\int\frac{x^6\cdot x}{\left(1-x^2\right)^5}\,dx$$

$$=-\frac{1}{2}\int\frac{(1-y)^3}{y^5}\,dy$$

$$=\frac{1}{2}\int\left\{\frac{y^3-1-3y^2+3y}{y^5}\right\}dy$$

$$=\frac{1}{2}\int\left\{y^{-2}-3y^{-3}+3y^{-4}-y^{-5}\right\}dy$$

$$=\frac{1}{2}\left\{-\frac{1}{y}+\frac{3}{2y^2}-\frac{1}{y^3}+\frac{1}{4y^4}\right\}+C$$

$$=\frac{1-4y+6y^2-4y^3}{8y^4}+C$$

$$\Rightarrow 1=\frac{1-4\left(1-x^2\right)+6\left(1-x^2\right)^2-4\left(1-x^2\right)^3}{8\left(1-x^2\right)^2}+C$$ ◀

(b) The denominator contains the term $(1-x^2)$. Think of a substitution that could cause the denominator to reduce to a single term: this substitution should be trignometric:

$$x=\sin\theta$$

$$dx=\cos\theta\,d\theta$$

$$I=\int\frac{x^7}{\left(1-x^2\right)^5}dx=\int\frac{\sin^7\theta\cos\theta}{\left(1-\sin^2\theta\right)^5}\,d\theta$$

$$=\int\frac{\sin^7\theta\cos\theta}{\cos^{10}\theta}\,d\theta$$

$$=\int\tan^7\theta\sec^2\theta\,d\theta$$

Notice now that the simple substitution $\tan\theta = y$ will reduce *I* to a simple integrable form:

$$\tan\theta = y$$

$$\Rightarrow \sec^2\theta d\theta = dy$$

$$I = \int \tan^7\theta \sec^2\theta d\theta = \int y^7 dy$$

$$= \frac{y^8}{8} + C$$

$$= \frac{\tan^8\theta}{8} + C$$

$$= \frac{x^8}{8\left(1-x^2\right)^4} + C \qquad (\because x = \sin\theta)$$

◀

There is one last point to be observed. The answers obtained by methods (a) and (b) might seem to be different from each other. However, verify that they are not! The answer obtained by (a) can be converted into the answer obtained by (b) by just a simple splitting of the constant of integration.

Example – 13

Evaluate $\int \frac{1}{\left(x^2+2x+2\right)^2} dx$.

Solution:

$$I = \int \frac{1}{\left(x^2+2x+2\right)^2} dx$$

$$= \int \frac{1}{\left((x+1)^2+1\right)^2} dx$$

The denominator can be reduced by the substitution

$$x+1 = \tan\theta$$

$$\Rightarrow (x+1)^2 + 1 = \tan^2\theta + 1 = \sec^2\theta$$

Also, $$dx = \sec^2\theta d\theta$$

$$\Rightarrow I = \int \frac{\sec^2\theta}{\sec^4\theta} d\theta$$

$$= \int \cos^2\theta d\theta$$

$$= \frac{1}{2}\int (1+\cos 2\theta) d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{1}{2}(\theta + \sin\theta\cos\theta) + C$$

To express the integral in terms of x, we use

$$x+1=\tan\theta$$

$$\Rightarrow \quad \theta=\tan^{-1}(x+1)$$

$$\Rightarrow \quad \sin\theta=\frac{(x+1)}{\sqrt{1+(x+1)^2}}$$

and

$$\cos\theta=\frac{1}{\sqrt{1+(x+1)^2}}$$

$$\Rightarrow \quad I=\frac{1}{2}\left\{\tan^{-1}(x+1)+\frac{x+1}{x^2+2x+2}\right\}+C$$

◀

Example – 14

Evaluate $\int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\cdot\frac{1}{x}dx$

Solution: A slight thought on the form of this expression will hint that a trignometric substitution might help; recall that both $(1-\cos\theta)$ and $(1+\cos\theta)$ can be reduced to single terms. Therefore, we use the substitution

$$\sqrt{x}=\cos\theta$$

$$\Rightarrow \quad x=\cos^2\theta$$

$$\Rightarrow \quad dx=-2\sin\theta\cos\theta d\theta$$

$$I=\int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\cdot\frac{1}{x}dx=\int\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\frac{1}{\cos^2\theta}\cdot(-2\sin\theta\cos\theta d\theta)$$

$$=-2\int\sqrt{\frac{2\sin^2(\theta/2)}{2\cos^2(\theta/2)}}\tan\theta d\theta$$

$$=-2\int\tan\left(\frac{\theta}{2}\right).\tan\theta d\theta$$

$$=-4\int\frac{\sin^2(\theta/2)}{\cos\theta}d\theta$$

$$=-2\int\frac{1-\cos\theta}{\cos\theta}d\theta$$

$$=-2\int(\sec\theta-1)d\theta$$

$$=-2\{\ln|\sec\theta+\tan\theta|-\theta\}+C$$

$$=-2\ln\left|\frac{1+\sin\theta}{\cos\theta}\right|+2\theta+C$$

$$=-2\ln\left(\frac{1+\sqrt{1-x}}{\sqrt{x}}\right)+2\cos^{-1}\sqrt{x}+C$$

◀

Example – 15

Evaluate $\int \frac{1}{x\sqrt{x^4-1}}dx$.

Solution: One possible way to reduce the term $\sqrt{x^4-1}$ is to use a trignometric substitution:

$$x^2 = \sec\theta$$

$$\Rightarrow \quad x^4 - 1 = \sec^2\theta - 1 = \tan^2\theta$$

$$\Rightarrow \quad 2xdx = \sec\theta\tan\theta d\theta$$

$$I = \int \frac{1}{x\sqrt{x^4-1}}dx = \int \frac{x}{x^2\sqrt{x^4-1}}dx$$

$$= \frac{1}{2}\int \frac{\sec\theta\tan\theta}{\sec\theta\tan\theta}d\theta$$

$$= \frac{1}{2}\int d\theta$$

$$= \frac{\theta}{2} + C$$

$$= \frac{1}{2}\sec^{-1}\left(x^2\right) + C$$

◀

Notice that for the five preceding examples, different substitution have been used in all the five. This shows that there is no hard-and-fast rule to do substitutions; you have to judge the most appropriate substitution by analyzing the expression of the function to be integrated.

Example – 16

Evaluate $\int \frac{\tan x}{a + b\tan^2 x}dx$

Solution: We first reduce this expression to another form involving sin and cos terms:

$$I = \int \frac{\sin x\cos x}{a\cos^2 x + b\sin^2 x}dx$$

If you observe the expression for I carefully, you will realise that a simple substitution in now possible:

$$a\cos^2 x + b\sin^2 x = y$$

$$\Rightarrow \quad (-2a\sin x\cos x + 2b\sin x\cos x)dx = dy$$

$$\Rightarrow \quad \sin x\cos xdx = \frac{dy}{2b-2a}$$

Thus we have obtained the numerator in terms of the derivative of the denominator:

$$I = \frac{1}{2(b-a)}\int \frac{dy}{y}$$

$$= \frac{\ln|y|}{2(b-a)} + C$$

$$= \frac{\ln|a\cos^2 x + b\sin^2 x|}{2(b-a)} + C$$

Example – 17

Evaluate $\int \frac{1}{\sqrt{\sin^3 x \sin(x+a)}}\,dx$

Solution: As in the previous example, we again have to modify the expression given to us, so that some substitution is possible. The first step that we could take is expand $\sin(x+a)$:

$$I = \int \frac{1}{\sqrt{\sin^3 x \sin(x+a)}}\,dx$$

$$= \int \frac{1}{\sqrt{\sin^3 x(\sin x \cos a + \cos x \sin a)}}\,dx$$

To proceed further, notice that there is a $\sin^3 x$ term in the denominator. What we now do is take out a common factor of $\sin x$ from the (inner) brackets so that $\sin^3 x$ becomes $\sin^4 x$:

$$I = \int \frac{1}{\sqrt{\sin^4 x\left(\cos a + \cot x \sin a\right)}}\,dx$$

$$= \int \frac{1}{\sin^2 x\sqrt{\cos a + \cot x \sin a}}\,dx$$

$$= \int \frac{\text{cosec}^2 x}{\sqrt{\cos a + \cot x \sin a}}\,dx$$

This is a form in which the numerator can be expressed as the derivative of the expression in the denominator.

Substitute $\quad \cos a + \cot x \sin a = t$

$\Rightarrow \quad -\sin a\,\text{cosec}^2 x\,dx = dt$

$$\Rightarrow \quad I = -\frac{1}{\sin a}\int \frac{dt}{\sqrt{t}}$$

$$= \frac{-2}{\sin a}\sqrt{t} + C$$

$$= \frac{-2\sqrt{\cos a + \cot x \sin a}}{\sin a} + C$$

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Example – 18

Evaluate the following integrals:

(a) $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx$ **(b)** $\int \frac{1}{a^2 + x^2}\,dx$ **(c)** $\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx$

(d) $\int \frac{1}{x^2 - a^2}\,dx$ **(e)** $\int \frac{1}{\sqrt{x^2 + a^2}}\,dx$ **(f)** $\int \frac{1}{\sqrt{x^2 - a^2}}\,dx$

Solution: This example is very important in the sense that the techniques subsequently described to evaluate these integrals can be used anywhere where such expressions are encountered.

Recall that the results to parts-(a), (b) and (c) have already been mentioned in the table titled '**Basic integration formulae**' on page -2. Also we have already seen (in examples 12, 13, 15), some of the integrals of these forms. Before starting with the solutions, consider the following table carefully which describes certain substitutions that can be used whenever expressions of the forms above are encountered.

	expression	can be reduced by the substitution
(1)	$a^2 - x^2$	$x = a\sin\theta$ or $x = a\cos\theta$
(2)	$a^2 + x^2$	$x = a\tan\theta$ or $x = a\cot\theta$
(3)	$x^2 - a^2$	$x = a\sec\theta$ or $x = a\,\text{cosec}\theta$

Verify that these substitutions will reduce the corresponding algebraic expressions to simpler trignometric expressions. We now proceed with the solutions:

(a) $I = \int \frac{1}{\sqrt{a^2 - x^2}}dx$

Substitute $x = a\sin\theta$

$\Rightarrow$ $dx = a\cos\theta d\theta$

and $a^2 - x^2 = a^2 - a^2\sin^2\theta$

$$= a^2\cos^2\theta$$

$$\Rightarrow \quad I = \int \frac{a\cos\theta}{a\cos\theta}d\theta$$

$$= \int d\theta$$

$$= \theta + C$$

$$= \sin^{-1}\frac{x}{a} + C \qquad ◀$$

Notice how convenient the integral became with the mentioned substitution.

(b) $I = \int \frac{1}{a^2 + x^2}dx$

Substitute $x = a\tan\theta$

$\Rightarrow$ $dx = a\sec^2\theta d\theta$

and $a^2 + x^2 = a^2 + a^2\tan^2\theta$

$$= a^2\sec^2\theta$$

$$\Rightarrow \quad I = \int \frac{a\sec^2\theta}{a^2\sec^2\theta}d\theta$$

$$= \frac{1}{a}d\theta$$

$$= \frac{\theta}{a} + C$$

$$= \frac{1}{a}\tan^{-1}\frac{x}{a} + C \qquad ◀$$

(c) $I = \int \frac{1}{x\sqrt{x^2 - a^2}} dx$

Substitute $x = a\sec\theta$

$\Rightarrow$ $dx = a\sec\theta\tan\theta d\theta$

and $x^2 - a^2 = a^2\sec^2\theta - a^2$

$= a^2\tan^2\theta$

$\Rightarrow$ $I = \int \frac{a\sec\theta\tan\theta}{a\sec\theta\sqrt{a^2\tan^2\theta}} d\theta$

$= \frac{1}{a}\int d\theta$

$= \frac{\theta}{a} + C$

$= \frac{1}{a}\sec^{-1}\frac{x}{a} + C$ ◀

(d) $I = \int \frac{1}{x^2 - a^2} dx$

Substitute $x = a\sec\theta$

$\Rightarrow$ $dx = a\sec\theta\tan\theta d\theta$

and $x^2 - a^2 = a^2\tan^2\theta$

$\Rightarrow$ $I = \int \frac{a\sec\theta\tan\theta}{a^2\tan^2\theta} d\theta$

$= \frac{1}{a}\int \operatorname{cosec}\theta \, d\theta$

$= \frac{1}{a}\ln\left|\operatorname{cosec}\theta - \cot\theta\right| + C$

$= \frac{1}{a}\ln\left|\frac{x}{\sqrt{x^2 - a^2}} - \frac{a}{\sqrt{x^2 - a^2}}\right| + C$

$= \frac{1}{a}\ln\left|\sqrt{\frac{x - a}{x + a}}\right| + C$

$= \frac{1}{2a}\ln\left|\frac{x - a}{x + a}\right| + C$ ◀

Instead of the substitution technique as described above, this integral could have alternatively been evaluate much more simply by a straightforward rearrangement of the expression:

$I = \int \frac{1}{x^2 - a^2} dx$

$= \frac{1}{2a}\int\left(\frac{1}{x - a} - \frac{1}{x + a}\right)dx$

$= \frac{1}{2a}\left(\ln(x - a) - \ln(x + a)\right) + C$

$= \frac{1}{2a}\ln\left(\frac{x - a}{x + a}\right) + C$ ◀

This again shows that there is no set rule to integration. You have to use your intuition to figure out the shortest possible route to the final answer.

(e) $I = \int \frac{1}{\sqrt{x^2 + a^2}} dx$

Substitute $x = a \tan\theta$

$\Rightarrow$ $dx = a \sec^2\theta d\theta$

and $x^2 + a^2 = a^2 \tan^2\theta + a^2$

$= a^2 \sec^2\theta$

$$\Rightarrow \quad I = \int \frac{a \sec^2\theta}{\sqrt{a^2 \sec^2\theta}} d\theta$$

$$= \int \sec\theta d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \ln\left|\sqrt{1 + \tan^2\theta} + \tan\theta\right| + C$$

$$= \ln\left|\frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a}\right| + C$$

$$= \ln\left|x + \sqrt{x^2 + a^2}\right| - \ln a + C$$

$$= \ln\left|x + \sqrt{x^2 + a^2}\right| + C'$$ ◀

(f) $I = \int \frac{1}{\sqrt{x^2 - a^2}} dx$

Substitute $x = a \sec\theta$

$\Rightarrow$ $dx = a \sec\theta \tan\theta d\theta$

and $x^2 - a^2 = a^2 \sec^2\theta - a^2$

$= a^2 \tan^2\theta$

$$\Rightarrow \quad I = \int \frac{a \sec\theta \tan\theta}{\sqrt{a^2 \tan^2\theta}} d\theta$$

$$= \int \sec\theta d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \ln\left|\sec\theta + \sqrt{\sec^2\theta - 1}\right| + C$$

$$= \ln\left|x + \sqrt{x^2 - a^2}\right| + C'$$ ◀

The expressions encountered in these six examples will widely be found elsewhere in this chapter too and therefore, you are advised to commit these six results to memory. If memorization is not possible for you, you should at least understand the techniques involved carefully so that you are quickly able to reproduce the answers whenever the need arises.