Section - 1 INTRODUCTION TO LIMITS

The concept of limits forms the basis of calculus and is a very powerful one. Both differential and integral calculus are based on this concept and as such, limits need to be studied in good detail.

This section contains a general, intuitive introduction to limits.

Consider a circle of radius *r*.

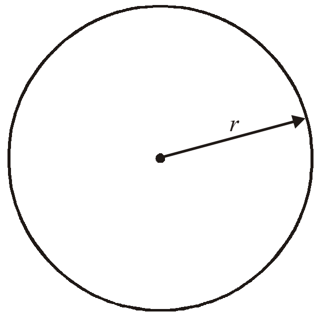


Fig - 1

We know that the area of this circle is πr^2. How?

The ancient Greeks derived this result using the concept of limits.

To see how, recall the definition of π.

$$\pi = \frac{length\,of\;\,circumference}{length\,of\;\,diameter}$$

$$\pi = \frac{c}{d} = \frac{c}{2r}$$

$$c = 2\pi r$$

With this definition in hand, the Greeks divided the circle as follows (like cutting a cake or a pie):

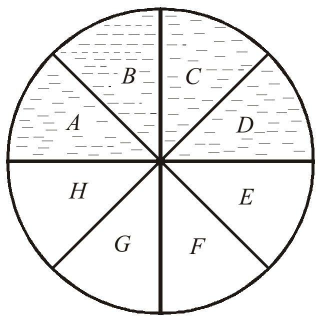
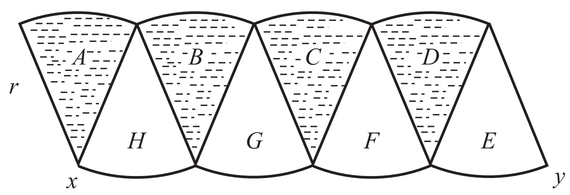


Fig - 2

Now they took the different pieces of this 'pie' and placed them as follows:



The total length of the four curved segments from x to y is πr.

Fig - 3

See what happens if the number of cuts are increased

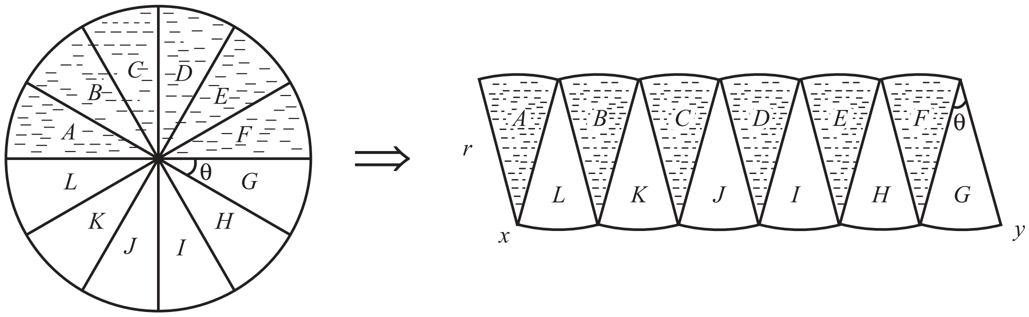


Fig - 4

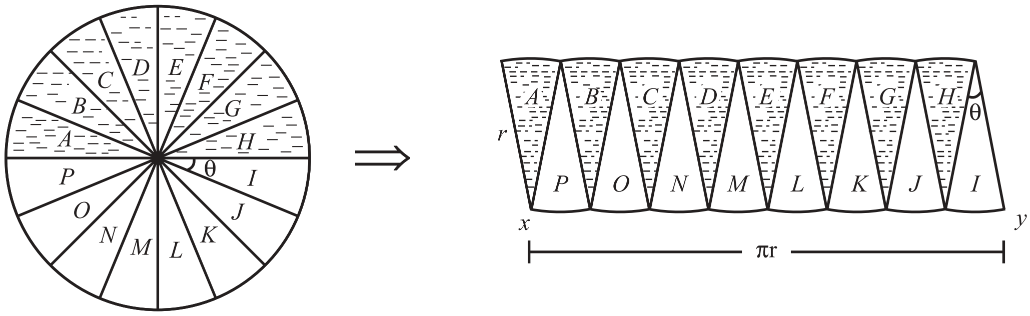


Fig - 5

The figure on the right side starts resembling a rectangle as we increase the number of cuts to the circle. The sequence of curves that joins x to y starts becoming more and more of a straight line with the same total length πr .

What happens as we increase the number of cuts indefinitely, or equivalently, we decrease θ indefinitely? The figure 'almost' becomes a rectangle, though never becoming a rectangle exactly. The area 'almost' becomes $\pi r \times r = \pi r^2$.

In the language of limits, we say that the figure *tends* to a rectangle or the area A tends to πr^2, or the *limiting value* of area is πr^2.

In standard terminology.

$$\lim_{\theta \to 0} A = \pi r^2$$

Hence, we see that a limit describes the behaviour of some quantity that depends on an independent variable, as that independent variable 'approaches' or 'comes close to' a particular value.

For example, how does $\dfrac{1}{x}$ behave when x becomes larger and larger? $\dfrac{1}{x}$ becomes smaller and smaller and 'tends' to 0.

We write this as

$$\lim_{x\to\infty}\frac{1}{x}=0$$

How does $\frac{1}{x}$ behave when x becomes smaller and smaller and approaches 0? $\frac{1}{x}$ obviously becomes larger and larger and 'tends' to infinity.

We write this as:

$$\lim_{x\to 0}\frac{1}{x}=\infty$$

The picture is not yet complete. In the example above, x can 'approach' 0 in two ways, either from the left hand side or from the right hand side:

$x \to 0^-$: approach is from left side of 0

$x \to 0^+$: approach is from right side of 0

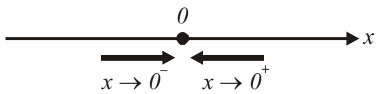


Fig - 6

How do we differentiate between the two possible approaches? Consider the graph of $f(x)=\frac{1}{x}$ carefully.

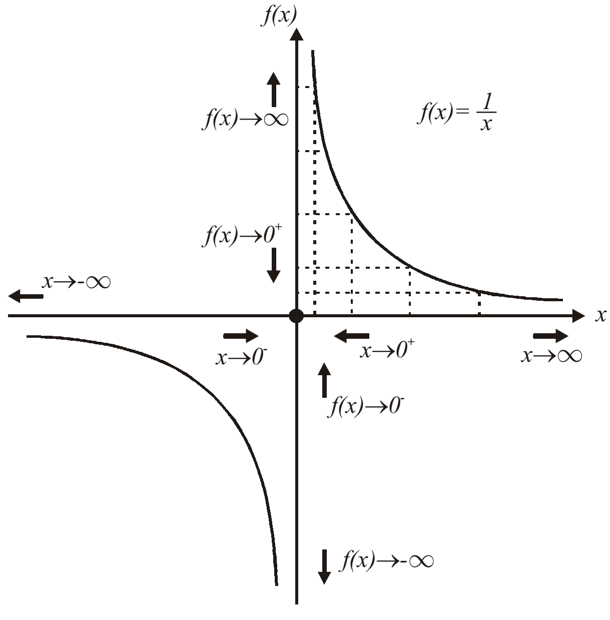


Fig - 7

As we can see in the graph above, as x increase in value or as $x \to \infty$, $f(x)$ decreases in value and approaches 0 (but it remains positive, or in other words, it approaches 0 from the positive side)

This can be written

$$\lim_{x\to\infty} f(x) = 0^+$$

Similarly, $\lim_{x\to 0^+} f(x) = +\infty$

What if x approaches 0, but from the left hand side $\left(x \to 0^-\right)$? From the graph, we see that as $x \to 0^-$, $\frac{1}{x}$ increases in magnitude but it also has a negative sign, that is $\frac{1}{x} \to -\infty$.

What if $x \to -\infty$? $\frac{1}{x}$ decreases in magnitude (approaches 0) but it still remains negative, that is, $\frac{1}{x}$ approaches 0 from the negative side or $\frac{1}{x} \to 0^-$

These concepts and results are summarized below:

(i)	$\lim_{x\to\infty} \frac{1}{x} = 0^+$	$\frac{1}{x}$ approaches 0 from the positive side
(ii)	$\lim_{x\to-\infty} \frac{1}{x} = 0^-$	$\frac{1}{x}$ approaches 0 from the negative side
(iii)	$\lim_{x\to 0^-} \frac{1}{x} = -\infty$	$\frac{1}{x}$ remains negative and increases in magnitude
(iv)	$\lim_{x\to 0^+} \frac{1}{x} = +\infty$	$\frac{1}{x}$ remains positive and increases in magnitude

Fig - 8

Lets consider another example now. We analyse the behaviour of $f(x) = [x]$, as x approaches 0.

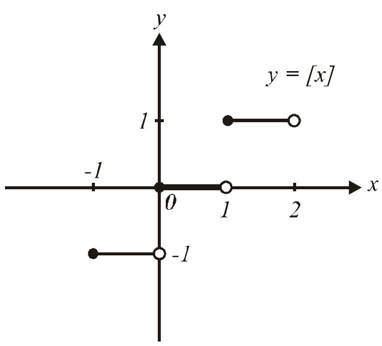


Fig - 9

What happens when x approaches 0 from the right hand side?

We see that $[x]$ remains 0.

What happens when x approaches 0 from the left hand side?

$[x]$ has a value -1.

Note that we are not talking about what value $[x]$ takes *at* $x = 0$. We are concerned with the behaviour of $[x]$ in the *neighbourhood* of $x = 0$, that is, to the immediate left and right of $x = 0$.

Hence, we have:

$$\lim_{x \to 0^+} [x] = 0$$

$$\lim_{x \to 0^-} [x] = -1$$

$$\left.\begin{aligned} &\lim_{x \to I^+} [x] = I \\ &\lim_{x \to I^-} [x] = I - 1 \end{aligned}\right\} I \text{ is any integer}$$

What about $f(x) = \{x\}$? This should be straightforward now:

$$\lim_{x \to 0^+} \{x\} = 0$$

$$\lim_{x \to 0^-} \{x\} = 1$$

$$\left.\begin{aligned} &\lim_{x \to I^+} \{x\} = 0 \\ &\lim_{x \to I^-} \{x\} = 1 \end{aligned}\right\} I \text{ is any integer}$$

We have now seen three different quantities regarding $f(x)$

(i)	The left hand limit (LHL) at $x = a$: $\lim\limits_{x \to a^-} f(x)$	Describes the behaviour of $f(x)$ to the immediate left of $x = a$
(ii)	The right hand limit (RHL) at $x = a$: $\lim\limits_{x \to a^+} f(x)$	Describes the behaviour of $f(x)$ to the immediate right of $x = a$
(iii)	The value of $f(x)$ at at $x = a$: $f(a)$	Gives the precise value that $f(x)$ takes at $x = a$

Fig - 10

Its possible that the value of the function at $x = a$ is undefined, and yet the LHL or RHL (or both) exist.

For example, consider $f(x) = \dfrac{x^2 - 1}{x - 1}$

$f(x)$ is clearly not defined at $x = 1$.

Every where else, $f(x)$ can be written in a simple form as

$$f(x) = \frac{(x-1)(x+1)}{(x-1)} = x+1$$

which is a straight line

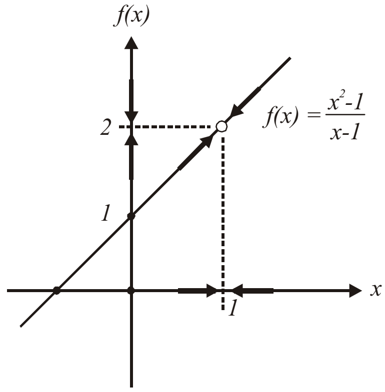


Fig - 11

This line has a hole at $x = 1$ because $f(x)$ is undefined there. In the neighbourhood of $x = 1\left(as\ x \to 1^+ \quad or \quad x \to 1^-\right)$,

$f(x)$ 'approaches' the value 2 (though it never becomes 2, because to become 2, x has to have the value 1, which is not possible). We see that

$$LHL = \lim_{x \to 1^-} f(x) = 2 = \lim_{x \to 1^+} f(x) = RHL$$

To emphasize once again, in evaluating a limit at $x = a$, we are not concerned with what value $f(x)$ assumes *at precisely* $x = a$; we are concerned with only how $f(x)$ behaves as x approaches or nearly becomes a, whether from the left hand or right hand side, giving rise to LHL and RHL respectively.

And finally, the **limit of $f(x)$ at $x = a$ is said to exist** if the function approaches the same value from both sides

$$LHL = RHL \quad at\ \ x = a$$

$$implies \lim_{x \to a} f(x)\ exists$$

$$and \quad \lim_{x \to a} f(x) = LHL = RHL$$

Note:

We have used the concept of ∞ (infinity) in the discussions above. Lets discuss this concept in some what more detail:

$\Rightarrow$ Infinity does not stand for any particular real number. In fact, it cannot be defined precisely. This is a deep concept. For any number you can think of, no matter how large, infinity is still larger. When we say that $x \to \infty$, we mean that x increase in an *unbounded* fashion, that is, becomes indefinitely large.

$\Rightarrow$ We cannot apply the normal rules of arithmetic to infinity. For example, saying that

$$\infty - \infty = 0 \qquad or \qquad \frac{\infty}{\infty} = 1$$

is absurd because such quantities are not defined.

$\Rightarrow$ It should be clear now that an expression like $x \to \infty$ defines a tendency (of unbounded increase)

Consider a fraction $f = \frac{Num}{Den}$

As $x \to$ a, if (Num) $\to$ (a finite number) and (Den) $\to \infty$, then f tends to become infinitesmally small or $f \to 0$.

Such cases are listed out below. Observe the details carefully

$$f = \frac{Num}{Den}$$

(i) $$\left.\begin{array}{l} \lim\limits_{x\to a}(Num) = finite \\ \lim\limits_{x\to a}(Den) = infinity \end{array}\right\} \Rightarrow \lim\limits_{x\to a} f = 0$$

(ii) $$\left.\begin{array}{l} \lim\limits_{x\to a}(Num) = infinity \\ \lim\limits_{x\to a}(Den) = finite \end{array}\right\} \Rightarrow \lim\limits_{x\to a} f = \infty$$

(iii) $$\left.\begin{array}{l} \lim\limits_{x\to a}(Num) = 0 \\ \lim\limits_{x\to a}(Den) = non\text{-}zero \end{array}\right\} \Rightarrow \lim\limits_{x\to a} f = 0$$

(iv) $$\left.\begin{array}{l} \lim\limits_{x\to a}(Num) = p \\ \lim\limits_{x\to a}(Den) = q \neq 0 \end{array}\right\} \Rightarrow \lim\limits_{x\to a} f = \frac{p}{q}$$

$\Rightarrow$ *Suppose that* $\lim\limits_{x\to a}(Num) = 0$ *and* $\lim\limits_{x\to a}(Den) = 0$

or $\lim\limits_{x\to a}(Num) = \infty$ *and* $\lim\limits_{x\to a}(Den) = \infty$,

then f is of the form:

$$f = \frac{\to 0}{\to 0} \quad or \quad \frac{\to \infty}{\to \infty}$$

The tendency (or limits) of such forms may or may not exist

For example, if (Num) = $x^2 - 1$ and (Den) = $x - 1$,

then $\lim\limits_{x\to 1}(Num) = 0$ *and* $\lim\limits_{x\to 1}(Den) = 0$, *so* that f is of the form $\frac{0}{0}$. But as we have seen earlier, the limit of f exists, as $x \to 1$

$$\lim_{x\to 1} f = \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1) = 2$$

Limits of the form $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, \quad \infty \times 0, \ 1^{\infty}$, 0^0 are called indeterminate forms. The limits of such forms may exist but it cannot be determined by simple observation (hence the name indeterminate). Such forms need to be reduced into determinate forms for which the limit can be determined. In the above case, for example, an indeterminate form $\left(\lim_{x\to 1} \frac{x^2-1}{x-1}\right)$ was reduced into a determinate form $\left(\lim_{x\to 1}(x+1)\right)$, whose limit can be determined by substitution of $x = 1$

Section - 2	SOME GENERAL RULES PERTAINING TO LIMITS

Now that we've had an intuitive introduction to limits, how do we go about evaluating limits for arbitrary functions?

For example, consider $f(x) = \frac{\sin x}{x}$. How does this function behave as $x \to 0$? (This, as we have seen in the previous section, is an indeterminate form).

Before moving any further, one should note that if a function $f(x)$ is determinate at $x = a$ and 'continuous' at $x = a$, implying that the graph of $f(x)$ has no 'break' at '$x = a$' (and in it's neighbourhood), the limit of $f(x)$ at $x = a$ can be evaluated by direct substitution. That is,

$$\lim_{x\to a} f(x) = f(a)$$

This will become obvious once we consider an example take $f(x) = x^2$.

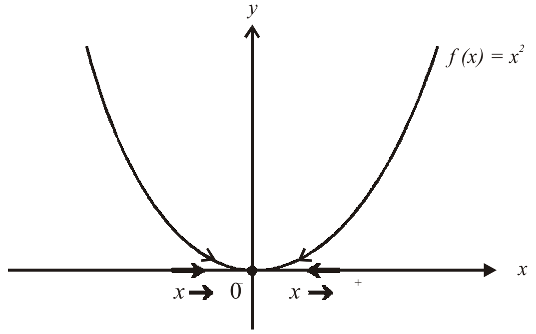


Fig - 12

Consider $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} x^2$.

From the graph, we see that since $f(x) = x^2$ is 'continuous' at $x = 0$ (it has no break), $f(x)$ will approach the same value from both sides, equal to $f(0)$

Hence, $\lim\limits_{x\to 0} x^2 = \lim\limits_{x\to 0^-} x^2 = LHL = \lim\limits_{x\to 0^+} x^2 = RHL = f(0) = 0$

Thus, any determinate limit can be evaluated by direct substitution of $x = a$. (The function needs to have no breaks at $x = a$)

For example:

$$\lim_{x\to 1}\left(x^2+2\right) = 1+2 = 3$$

$$\lim_{x\to 0}\left(x^3+4x+5\right) = 0+0+5 = 5$$

$$\lim_{x\to \pi} \sin x = \sin \pi = 0$$

What about $\lim\limits_{x\to 1}[x]$?

$$LHL = \lim_{x\to 1^-}[x] = 0$$

$$RHL = \lim_{x\to 1^+}[x] = 1$$

$$LHL \neq RHL$$

The limit does not exist ($f(x)$ is *discontinuous* at the point $x = 1$)

Note: Do not confuse continuity with existence of limits.

Existence of limit at $x = a$ implies $LHL = RHL$

Continuity of $f(x)$ at $x = a$ implies $LHL = RHL = f(a)$

We will soon study continuity in more detail.

The issue now is how to evaluate limits for indeterminate forms. This is the subject of the next section (where some standard indeterminate limits will be evaluated).

For now, observe some general rules pertaining to limits. These are more or less self explanatory.

Let $\lim\limits_{x\to a} f(x) = l$ and $\lim\limits_{x\to a} g(x) = m$.

If l, m exist, then:

(i) $\lim\limits_{x\to a}\{f(x)\pm g(x)\} = \left\{\lim\limits_{x\to a} f(x)\right\}\pm\left\{\lim\limits_{x\to a} g(x)\right\} = l \pm m$

(ii) $\lim\limits_{x\to a}\{f(x)\cdot g(x)\} = \left\{\lim\limits_{x\to a} f(x)\right\}\cdot\left\{\lim\limits_{x\to a} g(x)\right\} = l \cdot m$

(iii) $\lim\limits_{x\to a}\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{\left\{\lim\limits_{x\to a} f(x)\right\}}{\left\{\lim\limits_{x\to a} g(x)\right\}} = \dfrac{l}{m}$ $\quad$ *if* $m \neq 0$

(iv) $\lim\limits_{x\to a}\{K\, f(x)\} = K\left\{\lim\limits_{x\to a} f(x)\right\} = Kl$

(v) $\lim\limits_{x\to a}|f(x)| = \left|\lim\limits_{x\to a} f(x)\right| = |l|$

(vi) $\lim\limits_{x\to a}(f(x))^{g(x)} = \left(\lim\limits_{x\to a} f(x)\right)^{\left(\lim\limits_{x\to a} g(x)\right)} = l^m$, if the latter is defined.

(vii) $\lim\limits_{x\to a} f(g(x)) = f\left(\lim\limits_{x\to a} g(x)\right) = f(m)$

For example,

$\lim\limits_{x\to a}(\log f(x)) = \log\left(\lim\limits_{x\to a} f(x)\right)$

$\lim\limits_{x\to a} e^{f(x)} = e^{\lim\limits_{x\to a} f(x)}$

There needs to be a slight addition to this rule. Can you see what?
Consider $f(x) = [x]$ and $g(x) = x^3$

Then $\lim\limits_{x\to 0} f(g(x)) = \lim\limits_{x\to 0}[x^3]$.

Is this equal to $\left[\lim\limits_{x\to 0} x^3\right]$?

Does the former limit even exist?
Based on the answers to these questions, modify the rule above.

(viii) We now come to an interesting rule, called the **'Sandwich Rule'**. As the name suggests, this rule describes the behaviour of a function $f(x)$ 'sandwiched' between two different functions $g(x)$ and $h(x)$. At $x = a$, if $g(x)$ and $h(x)$, tend to the same value l, and $f(x)$ is sandwiched between $g(x)$ and $h(x)$, it will obviously also tend to the same value l.

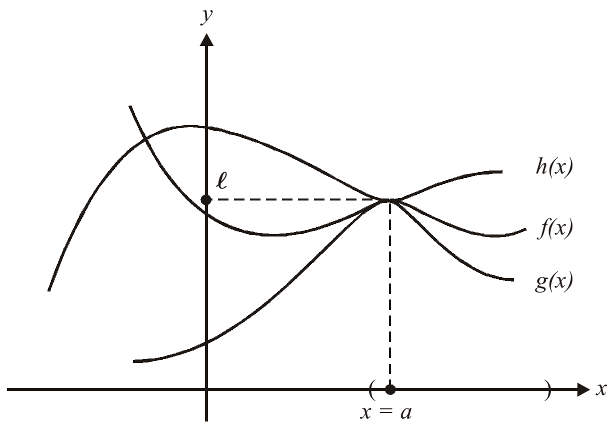


Fig - 13

For the graph above, in the neighbourhood of $x = a$, we see that the following always holds:

$$g(x) \le f(x) \le h(x)$$

or $f(x)$ is sandwiched between $g(x)$ and $h(x)$.

Also, since $\lim\limits_{x \to a} g(x) = \lim\limits_{x \to a} h(x) = l$,

we have $\lim\limits_{x \to a} f(x) = l$

This is the sandwich theorem.

Let us now move on to the evaluation of some standard indeterminate limits. A good grasp on these limits will help you evaluate almost all limits with ease.

Section - 3 SOME STANDARD LIMITS

(A) $\boxed{\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1; \ \lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1}$

Both the limits above are indeterminate, of the form $\dfrac{0}{0}$. We are discussing here a geometric interpretation of these limits. Consider a sector OAP of a unit circle as shown in the figure below.

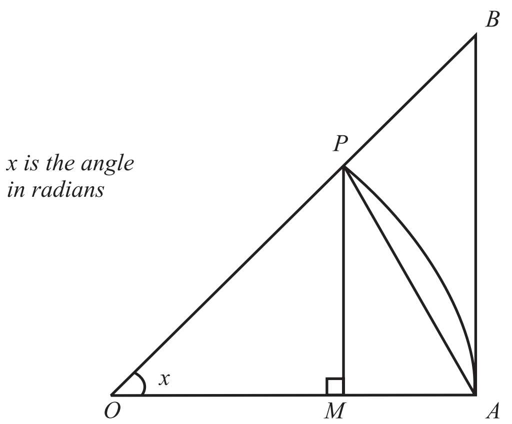


Fig - 14

$OA = OP = 1$ unit

$AB = OA \tan x = \tan x$

$\text{area}(\triangle OAP) = \dfrac{1}{2} \cdot OA \cdot OP \cdot \sin x$

$= \dfrac{1}{2} \sin x$

$\text{area (sector } OAP) = \dfrac{x}{2\pi} \times \pi r^2$

$= \dfrac{1}{2} x \ (\because r = 1)$

$\text{area}(\triangle OAB) = \dfrac{1}{2} \times OA \times AB$

$= \dfrac{1}{2} \tan x$

We see that

$$\text{area}(\Delta OAP) < \text{area}(\text{sector } OAP) < \text{area}(\Delta OAB)$$

or

$$\frac{1}{2}\sin x < \frac{1}{2}x < \frac{1}{2}\tan x$$

or

$$\frac{\sin x}{x} < 1 < \frac{\tan x}{x}$$

What happens as x decreases or as $x \to 0$?

We see that the difference between the three areas considered above tends to decrease;

$$area(\Delta OAP) \to area(\text{sector } OAP) \leftarrow area(\Delta OAB)$$

$$\Rightarrow \qquad \sin x \to x \qquad and \qquad x \to \tan x$$

$$\Rightarrow \qquad \frac{\sin x}{x} \to 1 \text{ (but remains less than 1)}$$

$$\frac{\tan x}{x} \to 1 \text{ (remains greater than 1)}$$

$$\Rightarrow \qquad \lim_{x\to 0} \frac{\sin x}{x} = 1^- \quad ; \quad \lim_{x\to 0} \frac{\tan x}{x} = 1^+$$

It is important to observe how any function approaches a limit.

For example, in the case above, as $x \to 0$, $\frac{\sin x}{x}$ approaches 1 from the left side while $\frac{\tan x}{x}$ approaches 1 from the right side. This makes a big difference.

Why?

Consider the limits below and you should understand:

$$\lim_{x\to 0}\left[\frac{\sin x}{x}\right] = 0 \quad ; \quad \lim_{x\to 0}\left[\frac{\tan x}{x}\right] = 1$$

(B) $\boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e}$

Consider the expression for $f(x) = \left(1+\frac{1}{x}\right)^x$. As x gets larger and larger or as $x \to \infty$, the base $\left(1+\frac{1}{x}\right)$ gets closer to 1 while the exponent (x), tends to infinity. Hence, this limit is of the indeterminate form 1^∞. Its very important to get a ‘feel’ that the value $\left(1+\frac{1}{x}\right)^x$ will converge to a fixed, definite value, as x increases. You should get this feel by looking at the table below.

x	$1+\frac{1}{x}$	$\left(1+\frac{1}{x}\right)^x$
1	2	2
10	1.1	2.5937...
100	1.01	2.7048...
1000	1.001	2.7169...
10000	1.0001	2.7181...
100000	1.00001	2.7182...

Fig - 15

We see that as x becomes larger, the term $\left(1+\frac{1}{x}\right)^x$ converges to some value (This can be proved)

This limiting value is denoted by e. e is an irrational number and its value is $e = 2.7182...$

Try to show that the limit of $\left(1+\frac{1}{x}\right)^x$ is bounded and lies between and 2 and 3 that is, $2 < e < 3$

(you can use the binomial theorem in conjunction with the sandwich theorem to prove this, by first proving it for an integral x)

The limit we have just seen is extremely important and will be widely used subsequently.

Note another important point:

If $f(x) \to 0 \quad as \quad x \to a, then$

$$\lim_{x\to a}\left(1+f(x)\right)^{1/f(x)} = \lim_{f(x)\to 0}\left(1+f(x)\right)^{1/f(x)}$$

$$\lim_{y\to 0}(1+y)^{1/y} = e \qquad \left(y = f(x)\right)$$

Hence, *any* limit of this form has the value e.

(C) $\boxed{\lim_{x\to 0}\frac{ln(1+x)}{x} = 1}$ $\quad (ln\ y \equiv log_e\ y)$

This limit is of the indeterminate form $\frac{0}{0}$.

We can easily evaluate this limit based on the previous limit.

$$\lim_{x\to 0}\frac{ln(1+x)}{x}$$

$$= \lim_{x\to 0}\frac{1}{x}ln(1+x)$$

$$= \lim_{x\to 0}ln(1+x)^{\frac{1}{x}} \qquad \text{(Property of log)}$$

$$= ln\left\{\lim_{x\to 0}(1+x)^{1/x}\right\}$$

$$= ln\ e = log_e e = 1$$

This limit can alternatively be evaluated by using the expansion series for $ln(1+x)$

$$ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ...\infty \qquad when\ |x| < 1$$

$$\Rightarrow \qquad \frac{ln(1+x)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} - ...\infty$$

$$\Rightarrow \qquad \lim_{x\to 0}\frac{ln(1+x)}{x} = 1 \quad \text{(All other terms involving } x \text{ tend to 0)}$$

(D)

$$\boxed{\lim_{x\to 0}\frac{log_a(1+x)}{x} = \frac{1}{ln\,a}}$$

This is just an obvious extension of the previous limit

$$\lim_{x\to 0}\frac{log_a(1+x)}{x} = \lim_{x\to 0}\frac{ln(1+x)}{(ln\,a)\cdot x}$$

$$= \frac{1}{ln\,a}\left(\lim_{x\to 0}\frac{ln(1+x)}{x}\right)$$

$$= \frac{1}{ln\,a}$$

We have used the following property of logarithms above:

$$log_a y = \frac{log_b y}{log_b a}$$

(E)

$$\boxed{\lim_{x\to 0}\frac{a^x - 1}{x} = ln\ a}$$

This is again an extension of the limits seen previously.

Let $a^x - 1 = t$. This gives $x = \log_a (1 + t)$

As $x \to 0,\ a^x \to 1$ and $a^x - 1 = t \to 0$

Hence, we have

$$= \lim_{x\to 0}\frac{a^x - 1}{x} = \lim_{t\to 0}\frac{t}{\log_a(1+t)}$$

$$= \lim_{t\to 0}\frac{1}{\left(\frac{\log_a(1+t)}{t}\right)} = \frac{1}{\lim\limits_{t\to 0}\left(\frac{\log_a(1+t)}{t}\right)}$$

$$= \frac{1}{1/\ln a} = \ln a.$$

Note that for $a = e$, this limit is 1.

(F) $\boxed{\lim_{x\to 1}\frac{x^m - 1}{x - 1} = m}$

When m is an integer, it is easy to see that the above relation holds because $x^m - 1$ can be expanded as $(x - 1)(x^{m-1} + x^{m-2} + \ldots\ldots + 1)$

For the general case, let $x = 1 + y$. As $x \to 1, y \to 0$.

Now we have $\lim\limits_{x\to 1}\frac{x^m - 1}{x - 1} = \lim\limits_{y\to 0}\frac{(1+y)^m - 1}{y}$

Now expand $(1 + y)^m$ using the Binomial theorem for a general index.

$$(1+y)^m = 1 + my + \frac{m(m-1)y^2}{2!} + \ldots$$

Hence, $\lim\limits_{y\to 0}\frac{(1+y)^m - 1}{y}$

$$= \lim_{y\to 0}\frac{my + \frac{m(m-1)}{2!}y^2 + \ldots}{y}$$

$$= \lim_{y\to 0}\left(m + \frac{m(m-1)}{2!}y + \ldots\right)$$

$= m$ (all other terms tend to 0)

(G) $$\boxed{\lim_{x\to a}\frac{x^m-a^m}{x-a}=m\,a^{m-1}}$$

This is an extension of the previous limit as follows:

$$\lim_{x\to a}\frac{x^m-a^m}{x-a}=\lim_{x\to a}\frac{a^m\left(\frac{x^m}{a^m}-1\right)}{a\left(\frac{x}{a}-1\right)}$$

$$= a^{m-1}\lim_{\left(\frac{x}{a}\to 1\right)}\frac{\left(\frac{x}{a}\right)^m-1}{\left(\frac{x}{a}\right)-1}=a^{m-1}\lim_{t\to 1}\frac{t^m-1}{t-1}\qquad\left(where\ t=\frac{x}{a}\right)$$

$$= m\,a^{\,m-1}$$

(H) $$\boxed{\lim_{x\to 0}\frac{\sin^{-1}x}{x}=\lim_{x\to 0}\frac{\tan^{-1}x}{x}=1}$$

We have seen the limits $\lim_{x\to 0}\frac{\sin x}{x}$ and $\lim_{x\to 0}\frac{\tan x}{x}$.

How can we use these limits to derive the limits that we require now?

Consider $\lim_{x\to 0}\frac{\sin^{-1}x}{x}$.

Substituting $\sin^{-1}x=t$ gives $x=\sin t$

Also, as $x\to 0, \sin^{-1}x\to 0$ or $t\to 0$.

Hence, our limit reduces to $\lim_{t\to 0}\frac{t}{\sin t}=\frac{1}{\lim_{t\to 0}\left(\frac{\sin t}{t}\right)}=1$

Similarly, $\lim_{x\to 0}\frac{\tan^{-1}x}{x}=1$

Evaluate $\lim_{x\to 0}\left[\frac{\sin^{-1}x}{x}\right]$ and $\lim_{x\to 0}\left[\frac{\tan^{-1}x}{x}\right]$

Having seen some important indeterminate limits, notice the following expansions that can be used to evaluate limits.

(a) $ln\left(1+x\right)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-...\infty \qquad |x|<1$

(b) $ln\left(1-x\right)=-x-\dfrac{x^2}{2}-\dfrac{x^3}{3}-...\infty \qquad |x|<1$

(c) $e^x=1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+...\infty \qquad x\,\varepsilon\,\mathbb{R}$

(d) $e^{-x}=1-\dfrac{x}{1!}+\dfrac{x^2}{2!}-\dfrac{x^3}{3!}+...\infty \qquad x\,\varepsilon\,\mathbb{R}$

(e) $sin\,x=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-...\infty$

(f) $cos\,x=1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-...\infty$

(g) $tan\,x=x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}+\dfrac{11x^7}{315}+...\infty$

(h) $\left(1+x\right)^n=1+nx+\dfrac{n\left(n-1\right)x^2}{2!}+...\infty \qquad \left\{\begin{matrix}|x|<1\\ and\; n\,\varepsilon\,Q\end{matrix}\right\}$

Note: We finally need to consider one last important form of limits, namely $\lim\limits_{x\to a}\left(f\left(x\right)\right)^{g(x)}$

(i) If $\lim\limits_{x\to a}f\left(x\right)=l>0,$ then we may write

$$\lim_{x\to a}\left(f\left(x\right)\right)^{g(x)}=\lim_{x\to a}e^{g(x)\ln f(x)}=e^{\lim\limits_{x\to a}g(x)\ln f(x)}$$

(ii) If $\lim\limits_{x\to a}f\left(x\right)=1$, then let $f(x)=1+h(x)$.

where $\lim\limits_{x\to a}h\left(x\right)=0$

Now $\lim\limits_{x\to a}\left(f\left(x\right)\right)^{g(x)}=\lim\limits_{x\to a}\left(1+h\left(x\right)\right)^{g(x)}$

$$=\lim_{x\to a}\left(1+h\left(x\right)\right)^{\frac{1}{h(x)}\cdot g(x)\cdot h(x)}=e^{\lim\limits_{x\to a}g(x)\cdot h(x)}$$

Section - 4 EVALUATION OF LIMITS

Now we discuss the various methods used in obtaining limits. Each method will be accompanied by some examples illustrating that method.

(A) DIRECT SUBSTITUTION

This already finds mention in Section - 2 where we saw that for a continuous function, the limit can be obtained by direct substitution.

This is because, by definition of a continuous function (at $x = a$):

$$LHL\left(at\ x=a\right)=RHL\left(at\ x=a\right)=f\left(a\right)$$

Hence, for example, all polynomial limits can be evaluated by direct substitution.

Some examples make all this clear:

(i) $\lim\limits_{x\to 1} x^3+1=2$

(ii) $\lim\limits_{x\to 2} 5x^2+3x+1=27$

(iii) $\lim\limits_{x\to -1} 4x^3+4=0$

(iv) $\lim\limits_{y\to 1} |y|+1=2$

(v) $\lim\limits_{x\to 5}\dfrac{5x^2+4}{2x+7}=\dfrac{129}{17}$

(vi) $\lim\limits_{x\to 8}\dfrac{x^3+1}{x+1}=\dfrac{513}{9}=57$ ◄

and so on

(B) FACTORIZATION

We saw an example of this method in evaluating $\lim\limits_{x\to 1}\dfrac{x^2-1}{x-1}$.

In such forms, the limit is indeterminate due to a certain factor occuring in the expression (For example, in the limit above, $(x - 1)$ occurs in both the numerator and denominator and makes the limit indeterminate, of the form $\dfrac{0}{0}$). Factorization leads to cancellation of that common factor and reduction of the limit to a determinate form.

(i) $\lim\limits_{x\to 1}\dfrac{x^3-1}{x-1}=\lim\limits_{x\to 1}\dfrac{(x-1)(x^2+x+1)}{x-1}=\lim\limits_{x\to 1}\left(x^2+x+1\right)=3$ ◄

Note that this limit is also of the form $\lim\limits_{x\to 1}\dfrac{x^m-1}{x-1}$ (whose limit is m)

(ii) $\lim\limits_{x\to 2}\dfrac{x^2-3x+2}{x^2-4}=\lim\limits_{x\to 2}\dfrac{(x-1)(x-2)}{(x-1)(x+2)}=\lim\limits_{x\to 2}\dfrac{x-1}{x+2}=\dfrac{1}{4}$ ◄

(iii) $$\lim_{x\to 0}\frac{(1+x)(1+2x)(1+3x)-1}{x}$$

$$=\lim_{x\to 0}\frac{1+6x+11x^2+6x^3-1}{x} = \lim_{x\to 0}\frac{6x+11x^2+6x^3}{x}$$

$$= \lim_{x\to 0}(6+11x+6x^2)=6$$ ◀

(iv) $$\lim_{x\to 1}\frac{x^4-3x+2}{x^5-4x+3}$$

We see that upon substitution of $x=1$, both the numerator and denominator become 0.
Hence, $(x-1)$ is a factor of both the numerator and denominator (Factor theorem)
Factorization leads to

$$\lim_{x\to 1}\frac{x^4-3x+2}{x^5-4x+3}=\lim_{x\to 1}\frac{\cancel{(x-1)}(x^3+x^2+x-2)}{\cancel{(x-1)}(x^4+x^3+x^2+x-3)} = \lim_{x\to 1}\frac{x^3+x^2+x-2}{x^4+x^3+x^2+x-3}$$

$$=\frac{1}{1}=1$$ ◀

(C) RATIONALIZATION

In this method, the rationalization of an indeterminate expression leads to determinate one. The following examples elaborate this method.

(i) $$= \lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}\left(\text{of the indeterminate form } \frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}\times\frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1}\times\frac{\sqrt{x^2+16}+4}{\sqrt{x^2+16}+4}\left(\begin{array}{l}\text{Rationalizing both the numerator}\\ \text{and the denominator}\end{array}\right)$$

$$= \lim_{x\to 0}\frac{\cancel{x^2}}{\cancel{x^2}}\times\frac{1}{\sqrt{x^2+1}+1}\times\frac{\sqrt{x^2+16}+4}{1}\quad(\text{A determinate form now!})$$

$$=\frac{8}{2}=4$$ ◀

(ii) $$\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2+1}\right)\ (\text{of the indeterminate form } \infty-\infty)$$

$$= \lim_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2+1}\right)\times\frac{\sqrt{x^2+x+1}+\sqrt{x^2+1}}{\sqrt{x^2+x+1}+\sqrt{x^2+1}}$$

$$= \lim_{x\to\infty}\frac{(x^2+x+1)-(x^2+1)}{\sqrt{x^2+x+1}+\sqrt{x^2+1}}$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x+1}+\sqrt{x^2+1}}$$

Rationalization has led us to another indeterminate form of $\frac{\infty}{\infty}$. However, it can easily be made determinate in the following manner:

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+x+1}+\sqrt{x^2+1}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1+\frac{1}{x^2}}} \qquad \left(\begin{array}{l}\textit{Divide the numerator} \\ \textit{and denominator by}\end{array}\right)$$

Now, as $x \to \infty$, $\frac{1}{x} \to 0$ and $\frac{1}{x^2} \to 0$

Hence, the limit above reduces to

$$\frac{1}{\sqrt{1+0+0}+\sqrt{1+0}} = \frac{1}{2}$$ ◀

(D) REDUCTION TO STANDARD FORMS

In this method, we try to reduce the given limit to one of the standard forms we studied in Section-3

(i) $\lim_{x\to 0} (1+\sin x)^{2\cot x}$

This limit is of the indeterminate form 1^{∞}.

$(as\ x \to 0, \sin x \to 0 \quad and \quad \cot x \to \infty)$

We proceed as follows:

$$\lim_{x\to 0} (1+\sin x)^{2\cot x} = \lim_{x\to 0} \left[\underbrace{(1+\sin x)^{\frac{1}{\sin x}}}_{\text{(This limit has the value e.)}} \right]^{2\cos x}$$

$$= e^{\lim_{x\to 0} 2\cos x} = e^2 (\cos x \to 1 \ \ as \ \ x \to 0)$$ ◀

(ii) $\lim_{x\to\frac{\pi}{2}} \frac{\cot x - \cos x}{(\pi - 2x)^3}$

This limit is of the indeterminate form $\frac{0}{0}$

Let $x = \frac{\pi}{2}+h$ so that as $x \to \frac{\pi}{2}, h \to 0.$

$$\lim_{x\to\frac{\pi}{2}} \frac{\cot x - \cos x}{(\pi - 2x)^3} = \lim_{h\to 0} \frac{\cot\left(\frac{\pi}{2}+h\right) - \cos\left(\frac{\pi}{2}+h\right)}{(-2h)^3}$$

$$= \lim_{h\to 0}\left(\frac{-\tan h+\sin h}{-8h^3}\right)$$

$$= \frac{1}{8}\lim_{h\to 0}\frac{-\sin h+\dfrac{\sin h}{\cos h}}{h^3}$$

$$= \frac{1}{8}\lim_{h\to 0}\frac{\sin h}{h}.\frac{1-\cos h}{h^2}.\frac{1}{\cos h}$$

$$= \frac{1}{8}\lim_{h\to 0}\frac{\sin h}{h}\cdot\frac{2\sin^2 h/2}{4(h/2)^2}\cdot\frac{1}{\cos h}$$

$$= \frac{1}{16}\lim_{h\to 0}\frac{\sin h}{h}\cdot\left(\frac{\sin h/2}{(h/2)}\right)^2\cdot\frac{1}{\cos h}$$

This expression now only contains the limits $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$ and $\lim\limits_{x\to 0}\cos x=1$

Hence, the final result is $\frac{1}{16}$ ◀

We will now see examples based on the methods discussed above. We urge you to first try out all these examples on your own before viewing the solutions.

SOLVED EXAMPLES

Example – 1

Evaluate the following limits

(a) $\lim\limits_{x\to 2}\left(\frac{2}{x(x-2)}-\frac{1}{x^2-3x+2}\right)$

(b) $\lim\limits_{n\to\infty}\left[(1+x)(1+x^2)(1+x^4).........(1+x^{2^n})\right]$ $\quad |x|<1$

(c) $\lim\limits_{x\to 3}\frac{x^3-7x^2+15x-9}{x^4-5x^2+27x-27}$

Solution: **(a)** This limit is of the indeterminate form $\infty - \infty$. Combining the two fractions in this limit should lead to a cancellation of the factor giving rise to this indeterminacy, i.e. $(x-2)$

$$\lim_{x\to 2}\left(\frac{2}{x(x-2)}-\frac{1}{x^2-3x+2}\right)$$

$$= \lim_{x\to 2}\frac{2(x-1)-x}{x(x-2)(x-1)} = \lim_{x\to 2}\frac{\cancel{(x-2)}}{x(x-1)\cancel{(x-2)}}$$

$$= \lim_{x\to 2}\frac{1}{x(x-1)} = \frac{1}{2}$$ ◀

(b) Before trying to solve this, try to feel that this expression will have a finite limit even though the number of factors being multiplied tends to infinity. This is because the successive factors become closer and closer to 1 and their 'contribution' to the final product becomes smaller and smaller

Now, to simplify this product, we multiply it by $\frac{1-x}{1-x}$. This is what happens:

$$\lim_{n\to\infty}\left[\frac{(1-x).\ (1+x)(1+x^2)(1+x^4)\ ......(1+x^{2^n})}{(1-x).}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(1-x^2)\ (1+x^2)\ (1+x^4)\ ......\ (1+x^{2^n})}{(1-x)}\right]$$

$$= \lim_{n\to 0}\left[\frac{(1-x^4)\ (1+x^4)\ ......\ (1+x^{2^n})}{(1-x)}\right]$$

$$\vdots$$

$$= \lim_{n\to\infty}\frac{1-x^{2^{n+1}}}{1+x}$$

Since $|x|<1, \lim\limits_{n\to\infty} x^{2^{n+1}} = 0$

Hence, the value of the limit is $\frac{1}{1-x}$ ◀

(c) The numerator and denominator both tend to 0 as $x\to 3$ because of the common factor $(x-3)$. Hence, factorization leads to :

$$\lim_{x\to 3}\frac{\cancel{(x-3)}(x^2-4x+3)}{\cancel{(x-3)}(x^2-2x^2-6x+9)} = \lim_{x\to 3}\frac{x^2-4x+3}{x^3-2x^2-6x+9}$$

There is still another common $(x-3)$ left in both the numerator and the denominator. Factorization again leads to

$$\lim_{x\to 3}\frac{(x-1)\cancel{(x-3)}}{\cancel{(x-3)}(x^2+x-3)} = \lim_{x\to 3}\frac{x-1}{x^2+x-3} = \frac{2}{9}$$ ◀

Example – 2

Evaluate the following limits:

(a) $\lim\limits_{x\to 4}\dfrac{\sqrt{1+2x}-3}{\sqrt{x}-2}$

(b) $\lim\limits_{x\to 1}\dfrac{x^2-\sqrt{x}}{\sqrt{x}-1}$

(c) $\lim\limits_{x\to\infty}\dfrac{\sqrt{x^2+1}-\sqrt[3]{x^2+1}}{\sqrt[4]{x^4+1}-\sqrt[5]{x^4+1}}$

(d) $\lim\limits_{x\to 1}\dfrac{(x+x^2+......+x^n)-n}{x-1}$

Solution: (a) This limit can evidently be solved by rationalising both the numerator and the denominator.

$$\lim_{x\to 4}\left\{\frac{\sqrt{1+2x}-3}{\sqrt{x}-2}\times\frac{\sqrt{1+2x}+3}{\sqrt{1+2x}+3}\times\frac{\sqrt{x}+2}{\sqrt{x}+2}\right\}$$

$$=\lim_{x\to 4}\left\{\frac{2x-8}{x-4}\times\frac{\sqrt{x}+2}{\sqrt{1+2x}+3}\right\}$$

$$=2\lim_{x\to 4}\frac{\sqrt{x}+2}{\sqrt{1+2x}+3}$$

$$=\frac{4}{3}$$

(b) This can be solved by rationalisation again.

$$\lim_{x\to 1}\left\{\frac{x^2-\sqrt{x}}{\sqrt{x}-1}\times\frac{x^2+\sqrt{x}}{x^2+\sqrt{x}}\times\frac{\sqrt{x}+1}{\sqrt{x}+1}\right\}$$

$$=\lim_{x\to 1}\left\{\frac{x^4-x}{x-1}\times\frac{\sqrt{x}+1}{x^2+\sqrt{x}}\right\}$$

$$=\lim_{x\to 1}\left\{x\left(x^2+x+1\right)\times\frac{\sqrt{x}+1}{x^2+\sqrt{x}}\right\}$$

$$=3$$

(c) This limit is of the indeterminate form $\dfrac{\infty-\infty}{\infty-\infty}$ (and look very complicated !)

However, division of both the numerator and denominator by x directly reduces the limit to a determinate form.

$$= \lim_{x\to\infty}\left\{\frac{\sqrt{\dfrac{x^2+1}{x^2}}-\sqrt[3]{\dfrac{x^2+1}{x^3}}}{\sqrt[4]{\dfrac{x^4+1}{x^4}}-\sqrt[5]{\dfrac{x^4+1}{x^5}}}\right\}$$

$$= \lim_{x\to\infty}\left\{\frac{\sqrt{1+\dfrac{1}{x^2}}-\boxed{\sqrt[3]{\dfrac{1}{x}+\dfrac{1}{x^3}}}}{\sqrt[4]{1+\dfrac{1}{x^4}}-\boxed{\sqrt[5]{\dfrac{1}{x}+\dfrac{1}{x^5}}}}\right\}$$

These two terms tend to 0 as $x\to\infty$

= 1 ◀

(d) since the denominator is $x-1$, we can get a hint that the numerator

$(x + x^2 + \ldots + x^n) - n$

can be written as

$(x-1) + (x^2-1) + \ldots + (x^n-1)$

so that

$$\lim_{x\to 1}\frac{\left(x+x^2+\ldots+x^n\right)-n}{x-1}=\lim_{x\to 1}\left\{\frac{x-1}{x-1}+\frac{x^2-1}{x-1}+\ldots+\frac{x^n-1}{x-1}\right\}$$

$= 1 + 2 + 3 + \ldots + n$

$$=\frac{n(n+1)}{2}$$ ◀

Example – 3

Find the values of a and b if $\lim\limits_{x\to\infty}\left\{\dfrac{x^2-1}{x+1}-(ax+b)\right\}=2$

Solution: The limit can be rearranged as

$$\lim_{x\to\infty}\left\{(x-1)-(ax+b)\right\}$$

$$=\lim_{x\to\infty}\left\{(1-a)x-(1+b)\right\}=2$$

Since the limit is finite, the coefficient of a has to be necessarily 0.

Therefore

$1-a=0 \quad \Rightarrow \quad a=1$

$-1-b=2 \quad \Rightarrow \quad b=-3$ ◀

Example – 4

Evaluate $\lim\limits_{n\to\infty}\dfrac{[x]+[2^2x]+[3^2x]+...+[n^2x]}{n^3}$ where [•] represents the greatest integer function.

Solution: One might say that since the terms in the numerator are all integral, the numerator is not continuous and hence the limit will not exist. However note first of all that the limit is on n, and the secondly, addition of a large number of integral terms in the numerator $(n\to\infty)$ would tend to 'overcome' or 'make negligible' the effect of fractional parts that would otherwise have been present had there been no greatest integer functions on any of the terms. This implies that whether I consider $([x]+[2^2x]+\ ....\ [n^2x])$, or $(x+2^2x+......n^2x)$, as n becomes larger and $\to\infty$, the difference between these two terms becomes negligible in comparison to their own magnitude. Hence, the limit in question is equivalent to

$$\lim_{n\to\infty}\frac{x+2^2x+3^2x+...n^2x}{n^3}$$

$$=\lim_{n\to\infty}\frac{x(1+2^2+3^2+...n^2)}{n^3}$$

$$=\frac{x}{6}\lim_{n\to\infty}\frac{n(n+1)(2n+1)}{n^3}$$

$$=\frac{x}{6}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\cdot\left(2+\frac{1}{n}\right)$$

$$=\frac{x}{3}$$ ◀

For students who like more rigor here is the proof of the above result using Sandwich theorem (In this proof, it will become clear that the effect of the fractional part's is negligible as $n\to\infty$

$$\left.\begin{aligned}x-1&<[x]\le x\\ 2^2x-1&<[2^2x]\le 2^2x\\ 3^2x-1&<[3^2x]\le 3^2x\\ n^2x-1&<[n^2x]\le n^2x\end{aligned}\right\}\quad \textit{By definition of the gretest integer function}$$

Addition of the these inequalities yields

$$(x+2^2x+3^2x+...+n^2x-n)<[x]+[2^2x]+...[n^2x]\le(x+2^2x+...n^2x)$$

Division by n^3 and application of $\lim\limits_{n\to\infty}$ on all three terms yields:

$$\lim_{n\to\infty}\left\{\frac{\frac{n(n+1)(2n+1)}{6}x - n}{n^3}\right\} < \lim_{n\to\infty}\left\{\frac{[x]+[2^2x]+...[n^2x]}{n^3}\right\} \le \lim_{n\to\infty}\left\{\frac{\frac{n(n+1)(2n+1)}{6}x}{n^3}\right\}$$

It is easy to that the left and right limits are both $\frac{x}{3}$, and hence the centre limit is also $\frac{x}{3}$. ◀

Example – 5

Evaluate the following limits:

(a) $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)(\cos x+\sin x)}$

(b) $\lim\limits_{x\to a}\dfrac{\cos\sqrt{x}-\cos\sqrt{a}}{x-a}$

(c) $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}$

(d) $\lim\limits_{n\to\infty}\left\{\cos\frac{x}{2}\cdot\cos\frac{x}{2^2}...\cos\frac{x}{2^n}\right\}$

Solution: Before proceeding to solve these limits, it should be mentioned that most limits can be evaluated in more than one manner. (In fact, L'Hospital's rule is a technique that can be used to solve most indeterminate limits: we will discuss it in detail in the topic of differentiation). In these examples and the ones that follow we will be discussing multiple ways for solving limits, wherever they are important.

(a) $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)(\cos x+\sin x)} = \lim\limits_{x\to\pi/4}\dfrac{1-\tan x}{\left(\frac{\pi}{4}-x\right)(1+\tan x)}$ (Dividing Num and Den by cos x)

Let $x=\frac{\pi}{4}+h,$ so that $\tan x=\frac{1+\tan h}{1-\tan h}$ and the limit above becomes

$$\lim_{h\to 0}\left\{\frac{1-\frac{1+\tan h}{1-\tan h}}{-h}\right\}\cdot\lim_{x\to\frac{\pi}{4}}\left\{\frac{1}{1+\tan x}\right\}$$

$$= \lim_{h\to 0}\left\{\frac{-2\tan h}{-h(1-\tan h)}\right\}\cdot\frac{1}{2} = \lim_{h\to 0}\left\{\frac{\tan h}{h}.\frac{1}{1-\tan h}\right\} = 1$$ ◀

Alternatively, we could proceed by multiplying the numerator and denominator of the original limit by *(cosx + sinx)*:

$$\lim_{x\to\frac{\pi}{4}}\left\{\frac{\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)(\cos x+\sin x)}\cdot\frac{\cos x+\sin x}{\cos x+\sin x}\right\}$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{\cos^2 x-\sin^2 x}{\left(\frac{\pi}{4}-x\right)(\cos x+\sin x)^2}$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{\cos 2x}{\left(\frac{\pi}{4}-x\right)}\cdot\lim_{x\to\frac{\pi}{4}}\frac{1}{(\cos x+\sin x)^2}$$

$$=\lim_{h\to 0}\frac{-\sin 2h}{-h}\cdot\frac{1}{2}\qquad\left(we\ \ let\ x=\frac{\pi}{4}+h\ \ so\,that\cos 2x=\cos\left(\frac{\pi}{2}+2h\right)=-\sin 2h\right)$$

$$=\lim_{h\to 0}\frac{\sin 2h}{2h}$$

= 1 ◀

(b) $\lim_{x\to a}\frac{\cos\sqrt{x}-\cos\sqrt{a}}{x-a}$

$$=\lim_{x\to a}\left\{\frac{-2\sin\left(\frac{\sqrt{x}+\sqrt{a}}{2}\right)\sin\left(\frac{\sqrt{x}-\sqrt{a}}{2}\right)}{\left(\sqrt{x}+\sqrt{a}\right)\left(\sqrt{x}-\sqrt{a}\right)}\right\}$$

{We used cos C – cos D = $-2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

$$=-\frac{\sin\sqrt{a}}{2\sqrt{a}}\cdot\lim_{x\to a}\left\{\frac{\sin\left(\frac{\sqrt{x}-\sqrt{a}}{2}\right)}{\left(\frac{\sqrt{x}-\sqrt{a}}{2}\right)}\right\}$$

$$=-\frac{\sin\sqrt{a}}{2\sqrt{a}}$$ ◀

(c) This limit can be solved very easily by using the expansion series for sin x :

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...\infty$$

Therefore, $\lim\limits_{x\to 0} \frac{x - \sin x}{x^3} = \lim\limits_{x\to 0} \left\{ \frac{\frac{x^3}{3!} - \frac{x^5}{5!} + ...\infty}{x^3} \right\}$

$$= \lim_{x\to 0} \left\{ \frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - ...\infty \right\}$$

$$= \frac{1}{3!} = \frac{1}{6}$$

Alternatively, we know that $\sin 3x = 3 \sin x - 4 \sin^3 x$

Hence, $3x - 3 \sin x = 3x - \sin 3x - 4 \sin^3 x$

$$L = \lim_{x\to 0} \frac{x - \sin x}{x^3} = \frac{1}{3} \lim_{x\to 0} \frac{3x - 3\sin x}{x^3}$$

$$= \frac{1}{3} \lim_{x\to 0} \left\{ \frac{3x - \sin 3x}{x^3} - \frac{4\sin^3 x}{x^3} \right\}$$

$$= \frac{1}{3} \lim_{x\to 0} \left\{ 27 \cdot \frac{(3x) - \sin(3x)}{(3x)^3} - \frac{4\sin^3 x}{x^3} \right\}$$

$$= \frac{1}{3} \cdot \left\{ 27 \cdot \lim_{\theta\to 0} \frac{\theta - \sin\theta}{\theta^3} - 4 \lim_{x\to 0} \left(\frac{\sin x}{x} \right)^3 \right\}$$

$$= 9L - \frac{4}{3}$$

Hence, $L = 9L - \frac{4}{3}$ or $L = \frac{1}{6}$

(d) As in Example - 1 Part - (b), we try to reduce this expression into a closed form by multiplying it with an appropriate factor as follows:

$$\lim_{n\to\infty} \left\{ \cos\frac{x}{2} \cdot \chi os \frac{x}{2^2} ... \cos\frac{x}{2^n} \cdot \frac{\sin\frac{x}{2^n}}{\sin\frac{x}{2^n}} \right\}$$

$$= \lim_{n\to\infty}\left\{\frac{\cos\frac{x}{2}\cdot\cos\frac{x}{2^2}\ldots\cos\frac{x}{2^{n-1}}\cdot\left(\overbrace{2\cos\frac{x}{2^n}\cdot\sin\frac{x}{2^n}}^{combine:\sin 2\theta=2\sin\theta\cos\theta}\right)}{2\sin\frac{x}{2^n}}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{\cos\frac{x}{2}\cdot\cos\frac{x}{2^2}\ldots\overbrace{\cos\frac{x}{2^{n-1}}\cdot\sin\frac{x}{2^{n-1}}}^{combine}}{2\sin\frac{x}{2^n}}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{\cos\frac{x}{2}\cdot\cos\frac{x}{2^2}\ldots\cos\frac{x}{2^{n-2}}\cdot\sin\frac{x}{2^{n-2}}}{2^2\cdot\sin\frac{x}{2^n}}\right\}$$

$$\vdots$$

$$= \lim_{n\to\infty}\frac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)}$$

$$= \lim_{n\to\infty}\frac{\sin x}{\dfrac{x\cdot\left(\sin\left(\frac{x}{2^n}\right)\right)}{\left(\frac{x}{2^n}\right)}} = \frac{\sin x}{x}\cdot\frac{1}{\lim\limits_{n\to\infty}\left\{\dfrac{\sin\left(\frac{x}{2^n}\right)}{\left(\frac{x}{2^n}\right)}\right\}}$$

Now, as $n\to\infty, \frac{x}{2^n}\to 0$ and hence

$$\lim_{n\to\infty}\left\{\frac{\sin\frac{x}{2^n}}{\frac{x}{2^n}}\right\} = 1$$

Therefore, our final result is $\dfrac{\sin x}{x}$ ◀

Example – 6

Evaluate the following limits:

(a) $\lim\limits_{x\to 0}\dfrac{e^{x^2}-\cos x}{x^2}$

(b) $\lim\limits_{x\to 0}\dfrac{x-\ln(1+x)}{x^2}$

(c) $\lim\limits_{x\to 0}\dfrac{(1+x)^{1/x}-e}{x}$

(d) $\lim\limits_{x\to 0}\dfrac{27^x-9^x-3^x+1}{\sqrt{2}-\sqrt{1+\cos x}}$

Solution : (a) The limit is of the indeterminate form $\dfrac{0}{0}$, but can be reduced into a combination of two standard limits as follows:

$$\lim_{x\to 0}\frac{e^{x^2}-\cos x}{x^2}=\lim_{x\to 0}\frac{e^{x^2}-1+1-\cos x}{x^2}$$

$$=\lim_{x\to 0}\left\{\frac{e^{x^2}-1}{x^2}+\frac{1-\cos x}{x^2}\right\}$$

$$=\lim_{x\to 0}\left\{\frac{e^{x^2}-1}{x^2}+\frac{2\sin^2 x/2}{4(x/2)^2}\right\}$$

$$=1+\frac{1}{2}=\frac{3}{2}$$ ◀

(b) $\ln(1+x)$ can be expanded as $x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\ldots$

Hence,

$$\lim_{x\to 0}\frac{x-\ln(1+x)}{x^2}=\lim_{x\to 0}\left\{\frac{\frac{x^2}{2}-\frac{x^3}{3}+\ldots}{x^2}\right\}$$

$$=\frac{1}{2}$$ ◀

(c) The numerator in this limits tends to 0 as $x\to 0$ because $\lim\limits_{x\to 0}(1+x)^{1/x}=e.$

Evaluating this limit will require a little artifice in the following manner:

$$\lim_{x\to 0}\left\{\frac{(1+x)^{1/x}-e}{x}\right\}=\lim_{x\to 0}\left\{\frac{e^{\frac{ln(1+x)}{x}}-e}{x}\right\}$$

$$= e\lim_{x\to 0}\left\{\frac{e^{\frac{ln(1+x)}{x}-1}-1}{x}\right\} \qquad \textit{Taking 'e' common out of the numerator}$$

Now as $x\to 0, \left\{\frac{ln(1+x)}{x}-1\right\}\to 0$ so that the numerator in the limit above is of the form e^h-1 where $h\to 0$.

What should we do now? Multiply and divide by h $\left(h=\frac{ln(1+x)}{x}-1\right)$.

We get

$$= e\lim_{x\to 0}\left[\frac{e^{\frac{ln(1+x)}{x}-1}-1}{\left\{\frac{ln(1+x)}{x}-1\right\}}\right]\cdot\left\{\frac{ln(1+x)}{x}-1\right\}\cdot\frac{1}{x}$$

$$= e\lim_{h\to 0}\left[\frac{e^h-1}{h}\right]\cdot\lim_{x\to 0}\left\{\frac{ln(1+x)-x}{x^2}\right\}$$

From the previous example, it follows that the second limit has the value $-\frac{1}{2}$.

Hence, the overall value for this limit is $-\frac{e}{2}$ ◀

(d) Factoring the numerator and rationalising the denominator gives.

$$\lim_{x\to 0}\left\{\frac{(9^x-1)(3^x-1)}{\sqrt{2}-\sqrt{1+cosx}}\cdot\frac{\sqrt{2}+\sqrt{1+cosx}}{\sqrt{2}+\sqrt{1+cosx}}\right\}$$

$$= \lim_{x\to 0}\frac{(9^x-1)}{x}\cdot\frac{(3^x-1)}{x}\cdot\frac{(\sqrt{2}+\sqrt{1+\cos x})}{1-\cos x}\cdot x^2$$

(We have multiplied and divided by x^2 above)

$$= \lim_{x\to 0}\frac{(9^x - 1)}{x}\cdot\lim_{x\to 0}\frac{(3^x - 1)}{x}\cdot\lim_{x\to 0}\left(\sqrt{2}+\sqrt{1+\cos x}\right)\cdot\lim_{x\to 0}\frac{2}{\dfrac{\sin^2 x/2}{(x/2)^2}}$$

$$= \ln 9.\ \ln 3.\ 2\sqrt{2}\ .\ 2$$

$$= 8\sqrt{2}\,(\ln 3)^2$$

◀

Example – 7

Evaluate the following limits:

(a) $\lim_{x\to 0}\left(\dfrac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$

(b) $\lim_{x\to 0}\left(\tan\left(\dfrac{\pi}{4}+x\right)\right)^{\frac{1}{x}}$

(c) $\lim_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{1/x^2}$

(d) $\lim_{x\to 0}(\cos x)^{1/\sin x}$

(e) $\lim_{x\to 0}(\cos x + a\sin b\,x)^{\frac{1}{x}}$

Solution: Notice that all the limits above are of the form $(f(x))^{g(x)}$ where $f(x)\to 1$ and $g(x)\to\infty$ that is, these limits are of the indeterminate form 1^{∞}.

In Section - 3, we saw how to evaluate such limits. Writing $f(x)$ as $(1 + h(x))$ reduces this limit to $e^{\lim_{x\to a} g(x)\cdot h(x)}$, where $h(x) = f(x) - 1$.

We will now directly apply this result to evaluate the limits above.

(a) $\lim_{x\to 0}\left(\dfrac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$

$$= e^{\lim_{x\to 0}\frac{1}{x}\cdot\left\{\frac{a^x+b^x+c^x}{3}-1\right\}}$$

$$= e^{\lim_{x\to 0}\frac{1}{3}\cdot\left\{\frac{(a^x-1)}{x}+\frac{(b^x-1)}{x}+\frac{(c^x-1)}{x}\right\}}$$

$$= e^{\frac{1}{3}((\ln a)+(\ln b)+(\ln c))} = abc^{1/3}$$

◀

(b) $\lim\limits_{x\to 0}\left(\tan\left(\dfrac{\pi}{4}+x\right)\right)^{\frac{1}{x}}$

$$= e^{\lim\limits_{x\to 0}\frac{1}{x}\cdot\left\{\tan\left(\frac{\pi}{4}+x\right)-1\right\}}$$

$$= e^{\lim\limits_{x\to 0}\left\{\frac{1}{x}\cdot\frac{2\tan x}{1-\tan x}\right\}} \qquad \left(\text{Using } \tan\left(\frac{\pi}{4}+x\right)=\frac{1+\tan x}{1-\tan x}\right)$$

$$= e^{2\lim\limits_{x\to 0}\left\{\frac{\tan x}{x}\cdot\frac{1}{1-\tan x}\right\}}$$

$$= e^{2}$$

(c) $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}$

$$= e^{\lim\limits_{x\to 0}\frac{1}{x^2}\cdot\left\{\frac{\sin x}{x}-1\right\}}$$

$$= e^{\lim\limits_{x\to 0}\left\{\frac{\sin x-x}{x^3}\right\}}$$

In example -5 Part -C, we considered the limit in the exponent above.

The value of this limit is therefore $e^{-1/6}$

(d) $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{\sin x}}$

$$= e^{\lim\limits_{x\to 0}\left\{\frac{1}{\sin x}\cdot(\cos x-1)\right\}}$$

$$= e^{\lim\limits_{x\to 0}\left(\frac{-2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)}$$

$$= e^{\lim\limits_{x\to 0}(-\tan x/2)} = e^{0} = 1.$$

(e) $\lim\limits_{x\to 0}(\cos x + a\sin bx)^{\frac{1}{x}}$

$$= e^{\lim\limits_{x\to 0}\frac{1}{x}\cdot\{\cos x+a\sin bx-1\}}$$

$$= e^{\lim\limits_{x\to 0}\left\{\frac{\cos x-1}{x}+ab\frac{\sin bx}{bx}\right\}}$$

$$= e^{\lim\limits_{x\to 0}\left\{\frac{-2\sin^2 x/2}{4(x/2)^2}\cdot x+ab\frac{\sin bx}{bx}\right\}}$$

$$= e^{\lim\limits_{x\to 0}\left\{-\frac{1}{2}\frac{\sin^2(x/2)}{(x/2)^2}\cdot x+ab\frac{\sin bx}{bx}\right\}}$$

$$= e^{ab}$$

Example – 8

Evaluate the following limits:

(a) $\lim\limits_{x \to 0} \sin\dfrac{1}{x}$

(b) $\lim\limits_{x \to 0} x \sin\dfrac{1}{x}$

(c) $\lim\limits_{x \to +\infty} \dfrac{\ln x}{x}$

(d) $\lim\limits_{x \to 0^+} x \ln x$

(e) $\lim\limits_{x \to +\infty} \dfrac{x^n}{n!}$

Solution: (a) Notice that as $x \to 0$, $\dfrac{1}{x} \to \infty$, that is , $\dfrac{1}{x}$ has no particular limit to which it converges. Hence $\sin\dfrac{1}{x}$ keeps oscillating between +1 and –1 as x becomes smaller and smaller, i.e., $x \to 0$. Therefore, the limit for this function does not exist. ◀

This is also clear from the graph (approximate) of $\sin\dfrac{1}{x}$ sketched below:

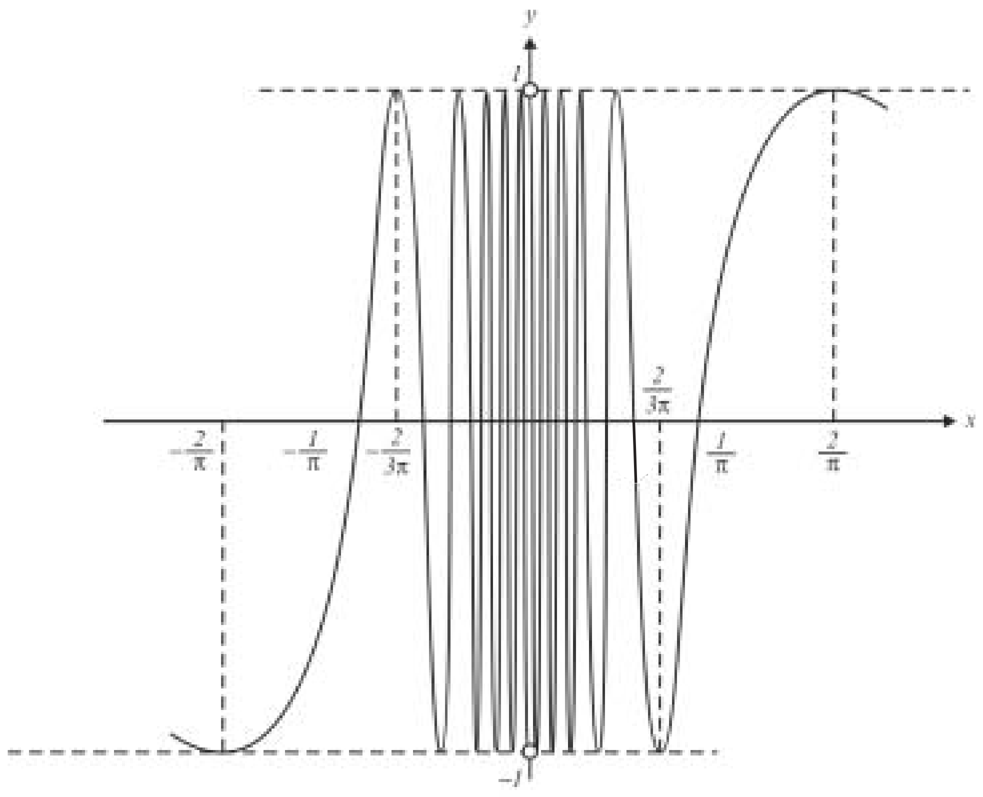


Fig. -16

(b) In this limit, in addition to $\sin\frac{1}{x}$, 'x' is also present. Thus, although $\sin\frac{1}{x}$ remains oscillating and does not approach any particular limit, it nevertheless remains somewhere between +1 and –1, and when it gets multiplied by x(where $x \to 0$), the whole product gets infinitesimally small. That is,

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0$$

Again, this is evident from the graph below:

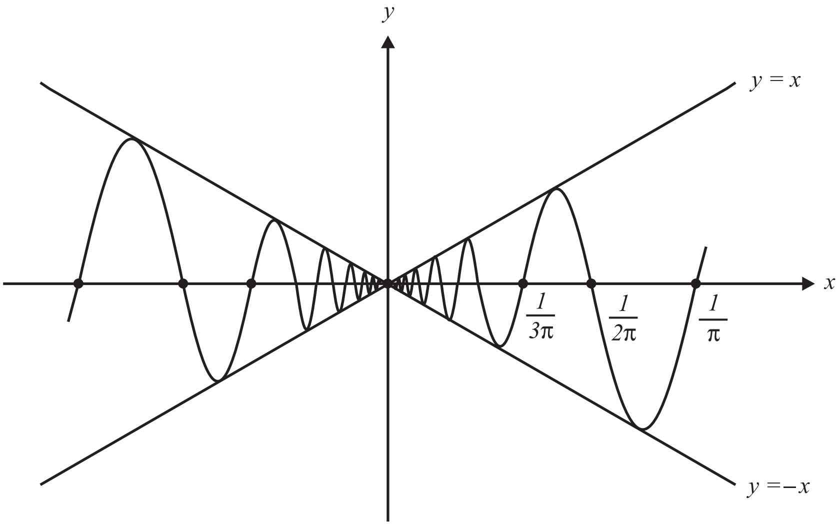


Fig. - 17

(c) This limit can be evaluated purely by observation as follow:

Although $\ln x$ and x are both tending to infinity, $\ln x$ increases *very slowly* as compared to x. For example, when $x = e^{10}$, $\ln x$ is just 10. When $x = e^{10000}$ (a very large number indeed !), $\ln x$ is just 10000.

Therefore, $\frac{\ln x}{x}$ decreases and becomes infinitesimally small as $x \to \infty$, i.e.,

$$\lim_{x\to\infty} \frac{\ln x}{x} = 0$$

(We can also use the LH rule to evaluate the limit above: this rule will be discussed later)

(d) Consider $x \ln x$.

As $x \to 0^+$, $\ln x \to -\infty$, so that this limit is of the indeterminate form $0 \times \infty$.

But as in parts (b) and (c), try to see that the product becomes infinitesimally small as $x \to 0$.

For example, at $x = e^{-10}$, $\ln x = -10$ and $x \ln x = \dfrac{-10}{e^{10}}$

At $x = e^{-1000}$, $x \ln x = \dfrac{-1000}{e^{1000}}$ (which is very very small)

Hence, here again, $\lim\limits_{x \to 0^+} x \ln x = 0$ ◀

(e) If $|x| < 1$, then as $n \to \infty$, $Num \to 0$ and $Den \to \infty$, so that the limit is 0.

For $|x| = 1$, also, the limit is obviously 0.

For $|x| > 1$ we write $\dfrac{x^n}{n!}$ as

$$\frac{x^n}{n!} = \frac{x}{1} \cdot \frac{x}{2} \cdot \frac{x}{3} \ldots \frac{x}{n}$$

Now, since x is finite, let N be the integer just less than or equal to x ; $N = [x]$

Hence, $\dfrac{x^n}{n!} = \dfrac{x}{1} \cdot \dfrac{x}{2} \ldots \dfrac{x}{N} \cdot \dfrac{x}{N+1} \cdot \dfrac{x}{N+2} \ldots \dfrac{x}{n}$

The product of the first N terms is finite; let it be equal to P.

Thus $\lim\limits_{n \to \infty} \dfrac{x^n}{n!} = P \lim\limits_{n \to \infty} \left\{ \dfrac{x}{N+1} \cdot \dfrac{x}{N+2} \ldots \dfrac{x}{n} \right\}$

The product inside the limit consists of all terms less than 1. Also successive terms become smaller and smaller and tend to 0 as $n \to \infty$.

Therefore, this product tends to 0 and hence the value of the overall limit is $P \times 0 = 0$

$\lim\limits_{n \to \infty} \dfrac{x^n}{n!} = 0$ ◀

EXERCISE

1. Evaluate the following limits

(i) $\lim\limits_{x\to a}\dfrac{x^m-a^m}{x^n-a^n}$

(ii) $\lim\limits_{n\to\infty}\dfrac{n!}{(n+1)!-n!}$

(iii) $\lim\limits_{n\to\infty}\dfrac{(n+1)!+(n+2)!}{(n+3)!}$

(iv) $\lim\limits_{x\to\infty}\dfrac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$

(v) $\lim\limits_{x\to\frac{\pi}{3}}\dfrac{\tan^3 x-3\tan x}{\cos\left(\dfrac{\pi}{6}+x\right)}$

(vi) $\lim\limits_{x\to 0}\dfrac{\tan x-x}{x^3}$

(vii) $\lim\limits_{x\to 0}\dfrac{1-\cos x\sqrt{\cos 2x}}{x^2}$

(viii) $\lim\limits_{x\to 0}\dfrac{1-\cos(1-\cos x)}{x^4}$

(ix) $\lim\limits_{x\to 0}\dfrac{\cos(\sin x)-\cos x}{x^4}$

(x) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x\sin x}-\sqrt{\cos x}}{\tan^2\dfrac{x}{2}}$

(xi) $\lim\limits_{x\to 0}\dfrac{\sin^{-1}x-\tan^{-1}x}{x^3}$

(xii) $\lim\limits_{x\to\infty} x\left\{\tan^{-1}\left(\dfrac{x+1}{x+4}\right)-\dfrac{\pi}{4}\right\}$

(xiii) $\lim\limits_{x\to\frac{\pi}{6}}\dfrac{\sin(x-\pi/6)}{\dfrac{\sqrt{3}}{2}-\cos x}$

(xiv) $\lim\limits_{\alpha\to\beta}\dfrac{\sin^2\alpha-\sin^2\beta}{\alpha^2-\beta^2}$

(xv) $\lim\limits_{x\to 0}\dfrac{x\tan 2x-2x\tan x}{(1-\cos 2x)^2}$

(xvi) $\lim\limits_{x\to 1}\dfrac{2^{x-1}-1}{\sin\pi x}$

(xvii) $\lim\limits_{x\to e}\dfrac{\ln x-1}{x-e}$

(xviii) $\lim\limits_{x\to a}\left(2-\dfrac{a}{x}\right)^{\frac{\tan\pi x}{2a}}$

(xix) $\lim\limits_{x\to\infty}\left(\sin\dfrac{1}{x}+\cos\dfrac{1}{x}\right)^x$

(xx) $\lim\limits_{x\to 0}\left(\dfrac{1^x+2^x+\ldots+n^x}{n}\right)^{a/x}$

(xxi) $\lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right)^{\frac{1}{x}}$

(xxii) $\lim\limits_{x\to\frac{\pi}{2}}\left(x\tan x-\dfrac{\pi}{2}\sec x\right)$

(xxiii) $\lim\limits_{x\to 0}\dfrac{2^x-1}{\sqrt{1+x}-1}$

(xxiv) $\lim\limits_{x\to 0}\dfrac{\sin(\pi\cos^2 x)}{x^2}$

(xxv) $\lim\limits_{x\to 0}\dfrac{6^x-2^x-3^x+1}{\sin^2 x}$

(xxvi) $\lim\limits_{x\to 1} x^{\cot\pi x}$

(xxvii) $\lim\limits_{x\to\infty} 5^{\frac{2x}{x+3}}$

(xxviii) $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cos x}{\sqrt[3]{(1-\sin x)^2}}$

(xxix) $\lim\limits_{x\to 0}\left(\dfrac{1}{x}-\dfrac{1}{\sin x}\right)$

(xxx) $\lim\limits_{x\to a}\dfrac{a\sin x-x\sin a}{ax^2-a^2x}$

2. Find a and b if $\lim\limits_{x\to 0}\dfrac{\sin 2x+a\sin x}{x^3}=b$

3. If $S_n=\sin\theta+\sin 2\theta+...+\sin n\theta$, prove that

$$\lim_{n\to\infty}\frac{S_1+S_2+...+S_n}{n}=\frac{1}{2}\cot\frac{\theta}{2}$$

You may use this relation: $S_n=\dfrac{\sin\dfrac{n\theta}{2}\cdot\sin\dfrac{(n+1)\theta}{2}}{\sin\dfrac{\theta}{2}}$

4. For $x > 0$, evaluate $\lim\limits_{x\to 0}\left\{(\sin x)^{\frac{1}{x}}+\left(\dfrac{1}{x}\right)^{\sin x}\right\}$

5. If $\lim\limits_{x\to 0}\dfrac{\{(a-n)nx-\tan x\}\sin(nx)}{x^2}=0$, find a.

6. Consider the curve $y = x^2$ for $0\le x\le 3$

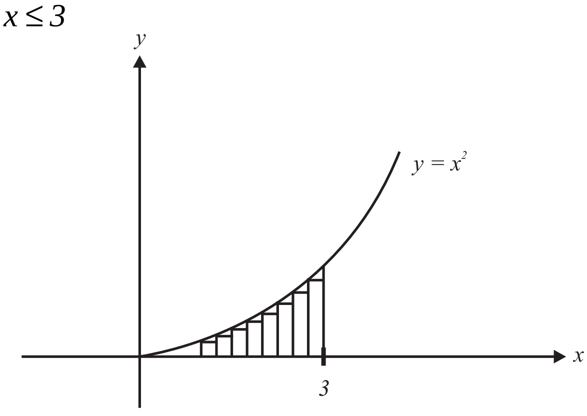


Fig - 18

Evaluate the shaded area.

(Hint: Divide this area approximately into n rectangles of width r, so that $r=\dfrac{3}{n}$. Now, the area of the i^{th} rectangle (counting from the left) can be written as a product of its height and its width r. Summing the areas of all the n rectangles gives an approximation to the shaded area above. Now taking the limit of this total area as $r\to 0$ or $n\to\infty$, we get the exact value of the shaded area. This technique is known as integration !)