2 Matrix Algebra

Section 2.1 Matrix Addition, Scalar Multiplication, and Transposition

In the study of systems of linear equations in Chapter 1, we found it convenient to manipulate the augmented matrix of the system. Our aim was to reduce it to row-echelon form (using elementary row operations) and hence to write down all solutions to the system. In the present chapter, we will consider matrices for their own sake, although some of the motivation comes from linear equations. This subject is quite old and was first studied systematically in 1858 by Arthur Cayley.[1]

A rectangular array of numbers is called a **matrix** (the plural is **matrices**), and the numbers are called the **entries** of the matrix. Matrices are usually denoted by uppercase letters: A, B, C, and so on. Hence,

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

are matrices. Clearly matrices come in various shapes depending on the number of **rows** and **columns**. For example, the matrix A shown has 2 rows and 3 columns. In general, a matrix with m rows and n columns is referred to as an $\boldsymbol{m \times n}$ **matrix** or as having **size** $\boldsymbol{m \times n}$. Thus matrices A, B, and C above have sizes 2×3, 2×2, and 3×1, respectively. A matrix of size $1 \times n$ is called a **row matrix**, whereas one of size $n \times 1$ is called a **column matrix**.

Each entry of a matrix is identified by the row and column in which it lies. The rows are numbered from the top down, and the columns are numbered from left to right. Then the $\boldsymbol{(i, j)}$**-entry** of a matrix is the number lying simultaneously in row i and column j. For example,

$$\text{The } (1, 2)\text{-entry of } \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \text{ is } -1$$

$$\text{The } (2, 3)\text{-entry of } \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 6 \end{bmatrix} \text{ is } 6$$

[1]Arthur Cayley (1821–1895) showed his mathematical talent early and graduated from Cambridge in 1842 as senior wrangler. With no employment in mathematics in view, he took legal training and worked as a lawyer while continuing to do mathematics, publishing nearly 300 papers in fourteen years. Finally, in 1863, he accepted the Sadlerian professorship at Cambridge and remained there for the rest of his life, valued for his administrative and teaching skills as well as for his scholarship. His mathematical achievements were of the first rank. In addition to originating matrix theory and the theory of determinants, he did fundamental work in group theory, in higher-dimensional geometry, and in the theory of invariants. He was one of the most prolific mathematicians of all time and produced 966 papers, filling thirteen volumes of 600 pages each.

A special notation has been devised for the entries of a matrix. If A is an $m \times n$ matrix, and if the (i, j)-entry of A is denoted as a_{ij}, then A is displayed as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

This is usually denoted simply as $A = [a_{ij}]$. Thus a_{ij} is the entry in row i and column j of A. For example, a 3×4 matrix in this notation is written

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

An $n \times n$ matrix A is called a **square matrix**. For a square matrix $A = [a_{ij}]$, the entries $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are said to lie on the **main diagonal** of the matrix A. Hence, the main diagonal extends from the upper left corner of A to the lower right corner (shaded in the following 3×3 matrix):

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

It is worth pointing out a convention regarding rows and columns: *Rows are mentioned before columns*. For example:

If a matrix has size $m \times n$, it has m rows and n columns.

If we speak of the (i, j)-entry of a matrix, it lies in row i and column j.

If an entry is denoted a_{ij}, the first subscript i refers to the row and the second subscript j to the column in which a_{ij} lies.

Two matrices A and B are called **equal** (written $A = B$) if and only if:

1. They have the same size.
2. Corresponding entries are equal.

If the entries of A and B are written in the form $A = [a_{ij}]$, $B = [b_{ij}]$, described earlier, then the second condition takes the following form:

$$[a_{ij}] = [b_{ij}] \quad \text{means} \quad a_{ij} = b_{ij} \text{ for all } i \text{ and } j.$$

Example 1

Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, discuss the possibility that $A = B$, $B = C$, $A = C$.

Solution

$A = B$ is impossible because A and B are of different sizes: A is 2×2 whereas B is 2×3. Similarly, $B = C$ is impossible. $A = C$ is possible provided that corresponding entries are equal: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ means $a = 1$, $b = 0$, $c = -1$, and $d = 2$.

Matrix Addition

If A and B are matrices of the same size, their **sum** $A + B$ is the matrix formed by adding corresponding entries. If $A = [a_{ij}]$ and $B = [b_{ij}]$, this takes the form

$$A + B = [a_{ij} + b_{ij}]$$

Note that addition is *not* defined for matrices of different sizes.

Example 2

If $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 6 \end{bmatrix}$, compute $A + B$.

Solution

$$A + B = \begin{bmatrix} 2+1 & 1+1 & 3-1 \\ -1+2 & 2+0 & 0+6 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 6 \end{bmatrix}.$$

Example 3

Find a, b, and c if $[a \;\; b \;\; c] + [c \;\; a \;\; b] = [3 \;\; 2 \;\; -1]$.

Solution

Add the matrices on the left side to obtain

$$[a+c \quad b+a \quad c+b] = [3 \quad 2 \quad -1]$$

Because corresponding entries must be equal, this gives three equations: $a + c = 3$, $b + a = 2$, and $c + b = -1$. Solving these yields $a = 3$, $b = -1$, $c = 0$.

If A, B, and C are any matrices *of the same size*, then

$$A + B = B + A \qquad \text{(commutative law)}$$
$$A + (B + C) = (A + B) + C \qquad \text{(associative law)}$$

In fact, if $A = [a_{ij}]$ and $B = [b_{ij}]$, then the (i, j)-entries of $A + B$ and $B + A$ are, respectively, $a_{ij} + b_{ij}$ and $b_{ij} + a_{ij}$. Since these are equal for all i and j, we get

$$A + B = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = B + A$$

The associative law is verified similarly.

The $m \times n$ matrix in which every entry is zero is called the **zero matrix** and is denoted as 0 (or 0_{mn} if it is important to emphasize the size). Hence,

$$0 + X = X$$

holds for all $m \times n$ matrices X. The **negative** of an $m \times n$ matrix A (written $-A$) is defined to be the $m \times n$ matrix obtained by multiplying each entry of A by -1. If $A = [a_{ij}]$, this becomes $-A = [-a_{ij}]$. Hence,

$$A + (-A) = 0$$

holds for all matrices A where, of course, 0 is the zero matrix of the same size as A.

A closely related notion is that of subtracting matrices. If A and B are two $m \times n$ matrices, their **difference** $A - B$ is defined by

$$A - B = A + (-B)$$

Note that if $A = [a_{ij}]$ and $B = [b_{ij}]$, then

$$A - B = [a_{ij}] + [-b_{ij}] = [a_{ij} - b_{ij}]$$

is the $m \times n$ matrix formed by *subtracting* corresponding entries.

Example 4

$A = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 2 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 0 & 6 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & 1 \end{bmatrix}$. Compute $-A$, $A - B$, and $A + B - C$.

Solution

$$-A = \begin{bmatrix} -3 & 1 & 0 \\ -1 & -2 & 4 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 3-1 & -1-(-1) & 0-1 \\ 1-(-2) & 2-0 & -4-6 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 2 & -10 \end{bmatrix}$$

$$A + B - C = \begin{bmatrix} 3+1-1 & -1-1-0 & 0+1+2 \\ 1-2-3 & 2+0-1 & -4+6-1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 3 \\ -4 & 1 & 1 \end{bmatrix}$$

Example 5

Solve $\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} + X = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, where X is a matrix.

Solution 1

X must be a 2×2 matrix. If $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, the equation reads

$$\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 3+x & 2+y \\ -1+z & 1+w \end{bmatrix}$$

The rule of matrix equality gives $1 = 3 + x$, $0 = 2 + y$, $-1 = -1 + z$, and $2 = 1 + w$. Thus, $X = \begin{bmatrix} -2 & -2 \\ 0 & 1 \end{bmatrix}$.

Solution 2

We solve a numerical equation $a + x = b$ by subtracting the number a from both sides to obtain $x = b - a$. This also works for matrices. To solve $\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} + X = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, simply subtract the matrix $\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ from both sides to get

$$X = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1-3 & 0-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 0 & 1 \end{bmatrix}$$

Of course, this is the same solution as obtained in Solution 1.

The two solutions in Example 5 are really different ways of doing the same thing. However, the first obtains four numerical equations, one for each entry, and solves them to get the four entries of X. The second solution solves the single matrix equation directly via matrix subtraction, and manipulation of entries comes in only at the end. The matrices themselves are manipulated. This ability to work with matrices as entities lies at the heart of matrix algebra.

It is important to note that the size of X in Example 5 was inferred *from the context*: X had to be a 2×2 matrix because otherwise the equation would not make sense. This type of situation occurs frequently; the sizes of matrices involved in some calculations are often determined by the context. For example, if

$$A + C = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 1 \end{bmatrix}$$

then A and C must be the same size (so that $A + C$ makes sense), and that size must be 2×3 (so that the sum is 2×3). For simplicity we shall often omit reference to such facts when they are clear from the context.

Scalar Multiplication

In Gaussian elimination, multiplying a row of a matrix by a number k means multiplying *every* entry of that row by k. More generally, if A is any matrix and k is any number, the **scalar multiple** kA is the matrix obtained from A by multiplying each entry of A by k. If $A = [a_{ij}]$, this is

$$kA = [ka_{ij}]$$

The term *scalar* arises here because the set of numbers from which the entries are drawn is usually referred to as the set of scalars. We have been using real numbers as scalars, but we could equally well have been using complex numbers.

Example 6

If $A = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 0 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \end{bmatrix}$, compute $5A$, $\frac{1}{2}B$, and $3A - 2B$.

Solution

$$5A = \begin{bmatrix} 15 & -5 & 20 \\ 10 & 0 & 30 \end{bmatrix}, \qquad \tfrac{1}{2}B = \begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & \frac{3}{2} & 1 \end{bmatrix}$$

$$3A - 2B = \begin{bmatrix} 9 & -3 & 12 \\ 6 & 0 & 18 \end{bmatrix} - \begin{bmatrix} 2 & 4 & -2 \\ 0 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 7 & -7 & 14 \\ 6 & -6 & 14 \end{bmatrix}$$

If A is any matrix, note that kA is the same size as A for all scalars k. We also have

$$0A = 0 \quad \text{and} \quad k0 = 0$$

because the zero matrix has every entry zero. In other words, $kA = 0$ if either $k = 0$ or $A = 0$. The converse of these properties is also true, as Example 7 shows.

Example 7

If $kA = 0$, show that either $k = 0$ or $A = 0$.

Solution

Write $A = [a_{ij}]$ so that $kA = 0$ means $ka_{ij} = 0$ for all i and j. If $k = 0$, there is nothing to do. If $k \neq 0$, then $ka_{ij} = 0$ implies that $a_{ij} = 0$ for all i and j; that is, $A = 0$.

For future reference, the basic properties of matrix addition and scalar multiplication are listed in Theorem 1.

Theorem 1

Let A, B, and C denote arbitrary $m \times n$ matrices where m and n are fixed. Let k and p denote arbitrary real numbers. Then

1. $A + B = B + A$.
2. $A + (B + C) = (A + B) + C$.
3. There is an $m \times n$ matrix 0, such that $0 + A = A$ for each A.
4. For each A there is an $m \times n$ matrix $-A$, such that $A + (-A) = 0$.
5. $k(A + B) = kA + kB$.
6. $(k + p)A = kA + pA$.
7. $(kp)A = k(pA)$.
8. $1A = A$.

Proof

Properties 1–4 were given previously. To check property 5, let $A = [a_{ij}]$ and $B = [b_{ij}]$ denote matrices of the same size. Then $A + B = [a_{ij} + b_{ij}]$, as before, so the (i, j)-entry of $k(A + B)$ is

$$k(a_{ij} + b_{ij}) = ka_{ij} + kb_{ij}$$

But this is just the (i, j)-entry of $kA + kB$, and it follows that $k(A + B) = kA + kB$. The other properties can be similarly verified; the details are left to the reader.

These properties enable us to do calculations with matrices in much the same way that numerical calculations are carried out. To begin, property 2 implies that the sum $(A + B) + C = A + (B + C)$ is the same no matter how it is formed and so is written as $A + B + C$. Similarly, the sum $A + B + C + D$ is independent of how it is formed; for example, it equals both $(A + B) + (C + D)$ and $A + [B + (C + D)]$. Furthermore, property 1 ensures that, for example, $B + D + A + C = A + B + C + D$. In other words, the *order* in which the matrices are added does not matter. A similar remark applies to sums of five (or more) matrices.

Properties 5 and 6 in Theorem 1 extend to sums of more than two terms. For example,

$$k(A + B + C) = kA + kB + kC$$
$$(k + p + m)A = kA = pA + mA$$

Similar observations hold for more than three summands. These facts, together with properties 7 and 8, enable us to simplify expressions by collecting like terms, expanding, and taking common factors in exactly the same way that algebraic expressions involving variables are manipulated. The following examples illustrate these techniques.

Example 8

Simplify $2(A + 3C) - 3(2C - B) - 3[2(2A + B - 4C) - 4(A - 2C)]$ where A, B, and C are all matrices of the same size.

Solution

The reduction proceeds as though A, B, and C were variables.

$$\begin{aligned} &2(A + 3C) - 3(2C - B) - 3[2(2A + B - 4C) - 4(A - 2C)] \\ &\quad = 2A + 6C - 6C + 3B - 3[4A + 2B - 8C - 4A + 8C] \\ &\quad = 2A + 3B - 3[2B] \\ &\quad = 2A - 3B \end{aligned}$$

Example 9

Find 1×3 matrices X and Y such that

$$\begin{aligned} X + 2Y &= [1 \quad 3 \quad -2] \\ X + Y &= [2 \quad 0 \quad 1] \end{aligned}$$

Solution

If we write $A = [1 \quad 3 \quad -2]$ and $B = [2 \quad 0 \quad 1]$, the equations become $X + 2Y = A$ and $X + Y = B$. The manipulations used to solve these equations when X, Y, A, and B represent numbers all apply in the present context. Hence, subtracting the second equation from the first gives $Y = A - B = [-1 \quad 3 \quad -3]$. Similarly, subtracting the first equation from twice the second gives $X = 2B - A = [3 \quad -3 \quad 4]$.

Transpose

Many results about a matrix A involve the *rows* of A, and the corresponding result for columns is derived in an analogous way, essentially by replacing the word *row* by the word *column* throughout. The following definition is made with such applications in mind. If A is an $m \times n$ matrix, the **transpose** of A, written A^T, is the $n \times m$ matrix whose rows are just the columns of A in the same order. In other words, the first row of A^T is the first column of A, the second row of A^T is the second column of A, and so on.

Example 10

Write down the transpose of each of the following matrices.

$$A = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \qquad B = [5 \quad 2 \quad 6] \qquad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

Solution

$$A^T = [1 \quad 3 \quad 2],\ B^T = \begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix},\ C^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}, \text{ and } D^T = D.$$

If $A = [a_{ij}]$ is a matrix, write $A^T = [b_{ij}]$. Then b_{ij} is the jth element of the ith row of A^T and so is the jth element of the ith *column* of A. This means $b_{ij} = a_{ji}$ so the definition of A^T can be stated as follows:

$$\text{If } A = [a_{ij}], \text{ then } A^T = [a_{ji}]$$

This is useful in verifying the following properties of transposition.

Theorem 2

Let A and B denote matrices of the same size, and let k denote a scalar.

1. If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix.
2. $(A^T)^T = A$.
3. $(kA)^T = kA^T$.
4. $(A + B)^T = A^T + B^T$.

Proof

We prove only property 3. If $A = [a_{ij}]$, then $kA = [ka_{ij}]$, so

$$(kA)^T = [ka_{ji}] = k[a_{ji}] = kA^T$$

which proves property 3.

The matrix D in Example 10 has the property that $D = D^T$. Such matrices are important; a matrix A is called **symmetric** if $A = A^T$. A symmetric matrix A is necessarily square (if A is $m \times n$, then A^T is $n \times m$, so $A = A^T$ forces $n = m$). The name comes from the fact that these matrices exhibit a symmetry about the main diagonal. That is, entries that are directly across the main diagonal from each other are equal. For example, $\begin{bmatrix} a & b & c \\ b' & d & e \\ c' & e' & f \end{bmatrix}$ is symmetric when $b = b'$, $c = c'$, and $e = e'$.

Example 11

If A and B are symmetric $n \times n$ matrices, show that $A + B$ is symmetric.

Solution

We have $A^T = A$ and $B^T = B$, so, by Theorem 2, $(A + B)^T = A^T + B^T = A + B$. Hence $A + B$ is symmetric.

Example 12

Suppose a square matrix A satisfies $A = 2A^T$. Show that necessarily $A = 0$.

Solution

If we iterate the given equation, Theorem 2 gives

$$A = 2A^T = 2[2A^T]^T = 2[2(A^T)^T] = 4A$$

Subtracting A from both sides gives $3A = 0$, so $A = \frac{1}{3}(3A) = \frac{1}{3}(0) = 0$.

Exercises 2.1

1. Find a, b, c, and d if

(a) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c - 3d & -d \\ 2a + d & a + b \end{bmatrix}$

♦(b) $\begin{bmatrix} a - b & b - c \\ c - d & d - a \end{bmatrix} = 2\begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}$

(c) $3\begin{bmatrix} a \\ b \end{bmatrix} + 2\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ♦(d) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & c \\ d & a \end{bmatrix}$

2. Compute the following:

(a) $\begin{bmatrix} 3 & 2 & 1 \\ 5 & 1 & 0 \end{bmatrix} - 5\begin{bmatrix} 3 & 0 & -2 \\ 1 & -1 & 2 \end{bmatrix}$

♦(b) $3\begin{bmatrix} 3 \\ -1 \end{bmatrix} - 5\begin{bmatrix} 6 \\ 2 \end{bmatrix} + 7\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix} + 3\begin{bmatrix} 2 & -3 \\ -1 & -2 \end{bmatrix}$

♦(d) $[3 \;\; -1 \;\; 2] - 2[9 \;\; 3 \;\; 4] + [3 \;\; 11 \;\; -6]$

(e) $\begin{bmatrix} 1 & -5 & 4 & 0 \\ 2 & 1 & 0 & 6 \end{bmatrix}^T$ ♦(f) $\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -4 \\ -2 & 4 & 0 \end{bmatrix}^T$

(g) $\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}^T$ ♦(h) $3\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}^T - 2\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

3. Let $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 1 & 4 \end{bmatrix}$, and $E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Compute the following (where possible).

(a) $3A - 2B$ ♦(b) $5C$ (c) $3E^T$
♦(d) $B + D$ (e) $4A^T - 3C$ ♦(f) $(A + C)^T$
(g) $2B - 3E$ ♦(h) $A - D$ (i) $(B - 2E)^T$

4. Find A if:

(a) $5A - \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = 3A - \begin{bmatrix} 5 & 2 \\ 6 & 1 \end{bmatrix}$

♦(b) $3A + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5A - 2\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

5. Find A in terms of B if:

(a) $A + B = 3A + 2B$

♦(b) $2A - B = 5(A + 2B)$

6. If X, Y, A, and B are matrices of the same size, solve the following equations to obtain X and Y in terms of A and B.

(a) $5X + 3Y = A$, $2X + Y = B$ ♦(b) $4X + 3Y = A$, $5X + 4Y = B$

7. Find all matrices X and Y such that:

(a) $3X - 2Y = [3 \ \ -1]$ ♦(b) $2X - 5Y = [1 \ \ 2]$

8. Simplify the following expressions where A, B, and C are matrices.

(a) $2[9(A - B) + 7(2B - A)]$
$-2[3(2B + A) - 2(A + 3B) - 5(A + B)]$

♦(b) $5[3(A - B + 2C) - 2(3C - B) - A]$
$+2[3(3A - B + C) + 2(B - 2A) - 2C]$

9. If A is any 2×2 matrix, show that:

(a) $A = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
for some numbers a, b, c, and d.

♦(b) $A = p\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + q\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + r\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + s\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
for some numbers p, q, r, and s.

10. Let $A = [1 \ \ 1 \ \ -1]$, $B = [0 \ \ 1 \ \ 2]$, and $C = [3 \ \ 0 \ \ 1]$. If $rA + sB + tC = 0$ for some scalars r, s, and t, show that necessarily $r = s = t = 0$.

11. (a) If $Q + A = A$ holds for every $m \times n$ matrix A, show that $Q = 0_{mn}$.

♦(b) If A is an $m \times n$ matrix and $A + A' = 0_{mn}$, show that $A' = -A$.

12. If A denotes an $m \times n$ matrix, show that $A = -A$ if and only if $A = 0$.

13. A square matrix is called a **diagonal** matrix if all the entries off the main diagonal are zero. If A and B are diagonal matrices, show that the following matrices are also diagonal.

(a) $A + B$ ♦(b) $A - B$ (c) kA for any number k

14. In each case determine all s and t such that the given matrix is symmetric:

(a) $\begin{bmatrix} 1 & s \\ -2 & t \end{bmatrix}$ ♦(b) $\begin{bmatrix} s & t \\ st & 1 \end{bmatrix}$

(c) $\begin{bmatrix} s & 2s & st \\ t & -1 & s \\ t & s^2 & s \end{bmatrix}$ ♦(d) $\begin{bmatrix} 2 & s & t \\ 2s & 0 & s+t \\ 3 & 3 & t \end{bmatrix}$

15. In each case find the matrix A.

(a) $\left(A + 3\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix}\right)^T = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 3 & 8 \end{bmatrix}$

♦(b) $\left(3A^T + 2\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)^T = \begin{bmatrix} 8 & 0 \\ 3 & 1 \end{bmatrix}$

(c) $(2A - 3[1 \ \ 2 \ \ 0])^T = 3A^T + [2 \ \ 1 \ \ -1]^T$

♦(d) $\left(2A^T - 5\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\right)^T = 4A - 9\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$

16. Let A and B be symmetric (of the same size). Show that each of the following is symmetric.

(a) $(A - B)$ ♦(b) kA for any scalar k

17. Show that $A + A^T$ is symmetric for *any* square matrix A.

18. A square matrix W is called **skew-symmetric** if $W^T = -W$. Let A be any square matrix.

(a) Show that $A - A^T$ is skew-symmetric.

(b) Find a symmetric matrix S and a skew-symmetric matrix W such that $A = S + W$.

♦(c) Show that S and W in part (b) are uniquely determined by A.

19. If W is skew-symmetric (Exercise 18), show that the entries on the main diagonal are zero.

20. Prove the following parts of Theorem 1.

(a) $(k + p)A = kA + pA$ ♦(b) $(kp)A = k(pA)$

21. Let $A, A_1, A_2, \ldots, A_n$ denote matrices of the same size. Use induction on n to verify the following extensions of properties 5 and 6 of Theorem 1.

(a) $k(A_1 + A_2 + \cdots + A_n) = kA_1 + kA_2 + \cdots + kA_n$ for any number k

(b) $(k_1 + k_2 + \cdots + k_n)A = k_1A + k_2A + \cdots + k_nA$ for any numbers $k_1, k_2, \ldots, k_n$

22. Let A be a square matrix. If $A = pB^T$ and $B = qA^T$ for some matrix B and numbers p and q, show that either $A = 0$ or $pq = 1$. [*Hint:* Examples 7 and 12.]

Section 2.2 Matrix Multiplication

Matrix multiplication is a little more complicated than matrix addition or scalar multiplication, but it is well worth the extra effort. It provides a new way to look at systems of linear equations as we shall see, and has a wide variety of other applications as well (for example, Sections 2.6 and 2.7).

If A is an $m \times n$ matrix and B is an $n \times k$ matrix, the **product** AB of A and B is the $m \times k$ matrix whose (i, j)-entry is computed as follows:

Multiply each entry of *row* i of A by the corresponding entry of *column* j of B, and add the results.

This is called the **dot product** of row i of A and column j of B.

Example 1

Compute the (1, 3)- and (2, 4)-entries of AB where

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix}$$

Then compute AB.

Solution

The (1, 3)-entry of AB is the dot product of row 1 of A and column 3 of B (high-lighted in the following display), computed by multiplying corresponding entries and adding the results.

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix} \quad (1, 3)\text{-entry} = 3\cdot 6 + (-1)\cdot 3 + 2\cdot 5 = 25$$

Similarly, the (2, 4) entry of AB involves row 2 of A and column 4 of B.

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix} \quad (2, 4)\text{-entry} = 0\cdot 0 + 1\cdot 4 + 4\cdot 8 = 36$$

Since A is 2×3 and B is 3×4, the product is 2×4.

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 25 & 12 \\ -4 & 2 & 23 & 36 \end{bmatrix}$$

Computing the (i, j)-entry of AB involves going *across* row i of A and *down* column j of B, multiplying corresponding entries, and adding the results. This requires that the rows of A and the columns of B be the same length. The following rule is a useful way to remember when the product of A and B can be formed and what the size of the product matrix is.

Rule

Suppose A and B have sizes $m \times n$ and $n' \times p$ respectively:

$$\begin{array}{ccc} & A & & B & \\ m & \times & n \quad n' & \times & p \end{array}$$

The product AB can be formed only when $n = n'$; in this case, the product matrix AB is of size $m \times p$. When this happens, we say that the product AB is **defined**.

Example 2

If $A = [1 \quad 3 \quad 2]$ and $B = \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix}$, compute A^2, AB, BA, and B^2 when they are defined.

Solution

Here, A is a 1×3 matrix and B is a 3×1 matrix, so A^2 and B^2 are not defined. However, the rule reads

$$\underset{1 \times 3}{A} \quad \underset{3 \times 1}{B} \qquad \text{and} \qquad \underset{3 \times 1}{B} \quad \underset{1 \times 3}{A}$$

so both AB and BA can be formed and these are 1×1 and 3×3 matrices, respectively.

$$AB = [1 \quad 3 \quad 2]\begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix} = [1 \cdot 5 + 3 \cdot 6 + 2 \cdot 4] = [31]$$

$$BA = \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix}[1 \quad 3 \quad 2] = \begin{bmatrix} 5 \cdot 1 & 5 \cdot 3 & 5 \cdot 2 \\ 6 \cdot 1 & 6 \cdot 3 & 6 \cdot 2 \\ 4 \cdot 1 & 4 \cdot 3 & 4 \cdot 2 \end{bmatrix} = \begin{bmatrix} 5 & 15 & 10 \\ 6 & 18 & 12 \\ 4 & 12 & 8 \end{bmatrix}$$

Unlike numerical multiplication, matrix products AB and BA *need not be equal*. In fact they need not even be the same size, as Example 2 shows. It turns out to be rare that $AB = BA$ (although it is by no means impossible). A and B are said to **commute** when this happens.

Example 3

Let $A = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$. Compute A^2, AB, and BA.

Solution

$A^2 = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $A^2 = 0$ can occur even if $A \neq 0$. Next,

$$AB = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 12 \\ 2 & -8 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ -6 & -9 \end{bmatrix}$$

Hence $AB \neq BA$, even though AB and BA are the same size.

The number 1 plays a neutral role in numerical multiplication in the sense that $1 \cdot a = a$ and $a \cdot 1 = a$ for all numbers a. An analogous role for matrix multiplication is played by square matrices of the following types:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and so on.}$$

In general, an **identity matrix** I is a *square matrix with 1's on the main diagonal and zeros elsewhere*. If it is important to stress the size of an $n \times n$ identity matrix, we shall denote it by I_n; however, these matrices are usually written simply as I. Identity matrices play a neutral role with respect to matrix multiplication in the sense that

$$AI = A \quad \text{and} \quad IB = B$$

whenever the products are defined.

Before proceeding, we must state the definition of matrix multiplication more formally. If $A = [a_{ij}]$ is $m \times n$ and $B = [b_{ij}]$ is $n \times p$, the ith row of A and the jth column of B are, respectively,

$$[a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}] \quad \text{and} \quad \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

Hence the (i, j)-entry of the product matrix AB is the dot product

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

where summation notation has been introduced for convenience.[2] This is useful in verifying facts about matrix multiplication.

Theorem 1

Assume that k is an arbitrary scalar and that A, B, and C are matrices of sizes such that the indicated operations can be performed.

1. $IA = A$, $BI = B$.
2. $A(BC) = (AB)C$.
3. $A(B + C) = AB + AC$; $A(B - C) = AB - AC$.
4. $(B + C)A = BA + CA$; $(B - C)A = BA - CA$.
5. $k(AB) = (kA)B = A(kB)$.
6. $(AB)^T = B^T A^T$.

Proof

We prove properties 3 and 6, leaving the rest as exercises.

Property 3. Write $A = [a_{ij}]$, $B = [b_{ij}]$, and $C = [c_{ij}]$ and assume that A is $m \times n$ and that B and C are $n \times p$. Then $B + C = [b_{ij} + c_{ij}]$, so the (i, j)-entry of $A(B + C)$ is

$$\sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj}) = \sum_{k=1}^{n}(a_{ik}b_{kj} + a_{ik}c_{kj}) = \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}$$

This is the (i, j)-entry of $AB + AC$ because the sums on the right are the (i, j)-entries of AB and AC, respectively. Hence $A(B + C) = AB + AC$; the other equation is proved similarly.

Property 6. Write $A^T = [a'_{ij}]$ and $B^T = [b'_{ij}]$, where $a'_{ij} = a_{ji}$ and $b'_{ij} = b_{ji}$. If B^T and A^T are $p \times n$ and $n \times m$, respectively, the (i, j)-entry of $B^T A^T$ is

$$\sum_{k=1}^{n} b'_{ik}a'_{kj} = \sum_{k=1}^{n} b_{ki}a_{jk} = \sum_{k=1}^{n} a_{jk}b_{ki}$$

This is the (j, i)-entry of AB—that is, the (i, j)-entry of $(AB)^T$. Hence $B^T A^T = (AB)^T$.

[2]Summation notation is a convenient shorthand way to write sums. For example, $a_1 + a_2 + a_3 + a_4 = \sum_{k=1}^{4} a_k$, $a_5x_5 + a_6x_6 + a_7x_7 = \sum_{i=5}^{7} a_i x_i$, and $1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \sum_{j=1}^{5} j^2$.

Property 2 in Theorem 1 asserts that the **associative law** $A(BC) = (AB)C$ holds for all matrices (if the products are defined). Hence, the product is the same no matter how it is formed and so is simply written ABC. This extends: The product $ABCD$ of four matrices can be formed several ways—for example, $(AB)(CD)$, $[A(BC)]D$, and $A[B(CD)]$—but property 2 implies that they are all equal and so are written simply as $ABCD$. A similar remark applies in general: Matrix products can be written unambiguously with no parentheses.

However, a note of caution about matrix multiplication *is* in order. The fact that AB and BA need *not* be equal means that the *order* of the factors is important in a product of matrices. For example, $ABCD$ and $ADCB$ may *not* be equal.

Warning

If the order of the factors in a product of matrices is changed, the product matrix may change (or may not exist).

Ignoring this warning is a source of many errors by students of linear algebra!

Properties 3 and 4 in Theorem 1 are called the **distributive laws**, and they extend to more than two terms. For example,

$$A(B - C + D - E) = AB - AC + AD - AE$$
$$(A + C - D)B = AB + CB - DB$$

Note again that the warning is in effect: For example, $A(B - C)$ need *not* equal $AB - CA$. Together with property 5 of Theorem 1, the distributive laws make possible a lot of simplification of matrix expressions.

Example 4

Simplify the expression $A(BC - CD) + A(C - B)D - AB(C - D)$.

Solution

$$\begin{aligned}&A(BC - CD) + A(C - B)D - AB(C - D)\\ &= ABC - ACD + (AC - AB)D - ABC + ABD\\ &= ABC - ACD + ACD - ABD - ABC + ABD\\ &= 0\end{aligned}$$

Examples 5 and 6 show how we can use the properties in Theorem 1 to deduce facts about matrix multiplication.

Example 5

Suppose that A, B, and C are $n \times n$ matrices and that both A and B commute with C; that is, $AC = CA$ and $BC = CB$. Show that AB commutes with C.

Solution

Showing that AB commutes with C means verifying that $(AB)C = C(AB)$. The computation uses property 2 of Theorem 1 several times, as well as the given facts that $AC = CA$ and $BC = CB$.

$$(AB)C = A(BC) = A(CB) = (AC)B = (CA)B = C(AB)$$

Example 6

Show that $AB = BA$ if and only if $(A - B)(A + B) = A^2 - B^2$.

Solution

Theorem 1 shows that the following always holds:

$$(A - B)(A + B) = A(A + B) - B(A + B) = A^2 + AB - BA - B^2 \qquad (*)$$

Hence if $AB = BA$, then $(A - B)(A + B) = A^2 - B^2$ follows. Conversely, if this last equation holds, then equation $(*)$ becomes

$$A^2 - B^2 = A^2 + AB - BA - B^2$$

This gives $0 = AB - BA$, and $AB = BA$ follows.

Matrices and Linear Equations

One of the most important motivations for matrix multiplication results from its close connection with systems of linear equations.

Example 7

Write the following system of linear equations as a single matrix equation.

$$\begin{aligned} 3x_1 - 2x_2 + x_3 &= b_1 \\ 2x_1 + x_2 - x_3 &= b_2 \end{aligned}$$

Solution

The two linear equations can be written as a single matrix equation as follows:

$$\begin{bmatrix} 3x_1 - 2x_2 + x_3 \\ 2x_1 + x_2 - x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

The matrix on the left can be factored as a product of matrices:

$$\begin{bmatrix} 3 & -2 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

If these matrices are denoted by A, X, and B, respectively, the system of equations becomes the matrix equation $AX = B$.

In the same way, consider *any* system of linear equations:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & b_m \end{array}$$

If $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$, these equations become the single matrix equation

$$AX = B$$

This is called the **matrix form** of the system of equations, and B is called the **constant matrix**. As in Section 1.1, A is called the **coefficient matrix** of the system, and a column matrix X_1 is called a **solution** to the system if $AX_1 = B$.

The matrix form is useful for formulating results about solutions of systems of linear equations. Given a system $AX = B$ there is a related system

$$AX = 0$$

called the **associated homogeneous system**. If X_1 is a solution to $AX = B$ and if X_0 is a solution to $AX = 0$, then $X_1 + X_0$ is a solution to $AX = B$. Indeed, $AX_1 = B$ and $AX_0 = 0$, so

$$A(X_1 + X_0) = AX_1 + AX_0 = B + 0 = B$$

This observation has a useful converse.

Theorem 2

Suppose X_1 is a particular solution to the system $AX = B$ of linear equations. Then every solution X_2 to $AX = B$ has the form

$$X_2 = X_0 + X_1$$

for some solution X_0 of the associated homogeneous system $AX = 0$.

Proof

Suppose that X_2 is *any* solution to $AX = B$ so that $AX_2 = B$. Write $X_0 = X_2 - X_1$. Then $X_2 = X_0 + X_1$, and we compute:

$$AX_0 = A(X_2 - X_1) = AX_2 - AX_1 = B - B = 0$$

Thus X_0 is a solution to the associated homogeneous system $AX = 0$.

The importance of Theorem 2 lies in the fact that sometimes a particular solution X_1 is easily found, and so the problem of finding all solutions is reduced to solving the associated homogeneous system.

Example 8

Express every solution to the following system as the sum of a specific solution plus a solution to the associated homogeneous system.

$$\begin{aligned} x - y - z &= 2 \\ 2x - y - 3z &= 6 \\ x \quad\quad - 2z &= 4 \end{aligned}$$

Solution

Gaussian elimination gives $x = 4 + 2t$, $y = 2 + t$, $z = t$, where t is arbitrary. Hence the general solution is

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4+2t \\ 2+t \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Thus $X_0 = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$ is a specific solution, and $X_1 = t\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ gives *all* solutions to the associated homogeneous system (do the Gaussian elimination with all the constants zero).

Theorem 2 focuses attention on homogeneous systems. In that case there is a convenient matrix form for the solutions that will be needed later. Example 9 provides an illustration.

Example 9

Solve the homogeneous system $AX = 0$ where

$$A = \begin{bmatrix} 1 & -2 & 3 & -2 \\ -3 & 6 & 1 & 0 \\ -2 & 4 & 4 & -2 \end{bmatrix}$$

Solution

The reduction of the augmented matrix to reduced form is

$$\begin{bmatrix} 1 & -2 & 3 & -2 \\ -3 & 6 & 1 & 0 \\ -2 & 4 & 4 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -\frac{1}{5} \\ 0 & 0 & 1 & -\frac{3}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so the solutions are $x_1 = 2s + \frac{1}{5}t,\ x_2 = s,\ x_3 = \frac{3}{5}t$ and $x_4 = t$ by Gaussian elimination. Hence we can write the general solution X in the matrix form

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s + \frac{1}{5}t \\ s \\ \frac{3}{5}t \\ t \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{1}{5} \\ 0 \\ \frac{3}{5} \\ 1 \end{bmatrix} = sX_1 + tX_2$$

where $X_1 = [2\ \ 1\ \ 0\ \ 0]^T$ and $X_2 = [\frac{1}{5}\ \ 0\ \ \frac{3}{5}\ \ 1]^T$ are particular solutions determined by the Gaussian algorithm.

The solutions X_1 and X_2 in Example 9 are called the **basic solutions** to the homogeneous system, and a solution of the form $sX_1 + tX_2$ is called a **linear combination** of the basic solutions X_1 and X_2.

In the same way, the Gaussian algorithm produces basic solutions to *every* homogeneous system $AX = 0$ (there are no basic solutions if there is only the trivial solution). Moreover, every solution is given by the algorithm as a linear combination of these basic solutions (as in Example 9). This proves most of

Theorem 3

Consider the homogeneous system $AX = 0$ in n variables where A has rank r. Then:

1. The system has exactly $n - r$ basic solutions.
2. Every solution is a linear combination of the basic solutions.

Proof

All that remains is to observe that there are exactly $n - r$ basic parameters by Theorem 2 §1.2, and hence $n - r$ basic solutions.

Example 10

Find the basic solutions of the system $AX = 0$ and express every solution as a linear combination of the basic solutions, where

$$A = \begin{bmatrix} 1 & -3 & 0 & 2 & 2 \\ -2 & 6 & 1 & 2 & -5 \\ 3 & -9 & -1 & 0 & 7 \\ -3 & 9 & 2 & 6 & -8 \end{bmatrix}$$

Solution
The reduction of the augmented matrix to reduced row-echelon form is

$$\begin{bmatrix} 1 & -3 & 0 & 2 & 2 & 0 \\ -2 & 6 & 1 & 2 & -5 & 0 \\ 3 & -9 & -1 & 0 & 7 & 0 \\ -3 & 9 & 2 & 6 & -8 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 6 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so the general solution is $x_1 = 3r - 2s - 2t$, $x_2 = r$, $x_3 = -6s + t$, $x_4 = s$, and $x_5 = t$ where r, s and t are parameters. In matrix form this is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3r - 2s - 2t \\ r \\ -6s + t \\ s \\ t \end{bmatrix} = r\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the basic solutions are $X_1 = [3 \ \ 1 \ \ 0 \ \ 0 \ \ 1]^T$, $X_2 = [-2 \ \ 0 \ \ -6 \ \ 1 \ \ 0]^T$, and $X_3 = [-2 \ \ 0 \ \ 1 \ \ 0 \ \ 1]^T$.

Block Multiplication

When forming matrix products YA and AX, it is often convenient to view the matrix A as a column of rows or as a row of columns. If A is $m \times n$, and if $R_1, R_2, \dots, R_m$ are the rows of A and $C_1, C_2, \dots, C_n$ are the columns, we write

$$A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix} \quad \text{and} \quad A = [C_1 \ \ C_2 \ \ \cdots \ \ C_n]$$

Then the definition of matrix multiplication shows that

$$AX = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix} X = \begin{bmatrix} R_1X \\ R_2X \\ \vdots \\ R_mX \end{bmatrix} \quad \text{and}$$

$$YA = Y[C_1 \ \ C_2 \ \ \cdots \ \ C_n] = [YC_1 \ \ YC_2 \ \ \cdots \ \ YC_n]$$

This gives AX in terms of its rows and YA in terms of its columns. In other words, the rows of AX are $R_1X, R_2X, \dots, R_mX$ and the columns of YA are $YC_1, YC_2, \dots, YC_n$.

Writing a matrix A as a row of columns, or as a column of rows, are special block decompositions of A, and these decompositions are related to matrix multiplication as the above results show. As another illustration, write the 2×3 matrix A as $A = [C_1 \ \ C_2 \ \ C_3]$ where C_1, C_2 and C_3 denote the columns of A, and let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be a column. If we (rather naively) view A as a row matrix, the product AX becomes

$$AX = [C_1 \ \ C_2 \ \ C_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1C_1 + x_2C_2 + x_3C_3$$

The amazing thing is that this is true! Indeed, if we write $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$ so that $C_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$, $C_2 = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$, and $C_3 = \begin{bmatrix} a_3 \\ b_3 \end{bmatrix}$, then

$$\begin{aligned} AX &= \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_1x_1 + a_2x_2 + a_3x_3 \\ b_1x_1 + b_2x_2 + b_3x_3 \end{bmatrix} \\ &= x_1 \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + x_2 \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} + x_3 \begin{bmatrix} a_3 \\ b_3 \end{bmatrix} \\ &= x_1C_1 + x_2C_2 + x_3C_3 \end{aligned}$$

This holds in general, and the result will be used several times in this book.

Theorem 4

Let $A = [C_1 \ \ C_2 \ \ \cdots \ \ C_n]$ be an $m \times n$ matrix with columns $C_1, C_2, \ldots, C_n$. If $X = [x_1 \ \ x_2 \ \ \cdots \ \ x_n]^T$ is any column, then

$$AX = [C_1 \ \ C_2 \ \ \cdots \ \ C_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1C_1 + x_2C_2 + \cdots + x_nC_n$$

These are special cases of a more general way of looking at matrices that, among its other uses, can greatly simplify matrix multiplications. The idea is to partition a matrix A into smaller matrices (called **blocks**) by inserting vertical lines between the columns and horizontal lines between the rows.[3]

As an example, consider the matrices

$$A = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 2 & -1 & 4 & 2 & 1 \\ 3 & 1 & -1 & 7 & 5 \end{array}\right] = \begin{bmatrix} I_2 & 0_{23} \\ P & Q \end{bmatrix} \quad \text{and} \quad B = \left[\begin{array}{cc} 4 & -2 \\ 5 & 6 \\ \hline 7 & 3 \\ -1 & 0 \\ 1 & 6 \end{array}\right] = \begin{bmatrix} X \\ Y \end{bmatrix}$$

where the blocks have been labeled as indicated. This is a natural way to think of A in view of the blocks I_2 and 0_{23} that occur. This notation is particularly useful when we are multiplying the matrices A and B because the product AB can be computed in block form as follows:

$$AB = \begin{bmatrix} I & 0 \\ P & Q \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} IX + 0y \\ PX + Qy \end{bmatrix} = \begin{bmatrix} X \\ PX + QY \end{bmatrix} = \left[\begin{array}{cc} 4 & -2 \\ 5 & 6 \\ \hline 30 & 8 \\ 8 & 27 \end{array}\right]$$

This is easily checked to be the product AB, computed in the conventional manner.

In other words, *we can compute the product AB by ordinary matrix multiplication, using blocks as entries*. The only requirement is that the blocks be **compatible**.

[3] We have been doing this already with the augmented matrices arising from systems of linear equations.

That is, *the sizes of the blocks must be such that all (matrix) products of blocks that occur make sense*. This means that the number of columns in each block of A must equal the number of rows in the corresponding block of B.

Block Multiplication

If matrices A and B are partitioned compatibly into blocks, the product AB can be computed by matrix multiplication using blocks as entries.

We omit the proof and instead give one more example of block multiplication that will be used below.

Theorem 5

Suppose that matrices $A = \begin{bmatrix} B & X \\ 0 & C \end{bmatrix}$ and $A_1 = \begin{bmatrix} B_1 & X_1 \\ 0 & C_1 \end{bmatrix}$ are partitioned as shown where B and B_1 are square matrices of the same size, and C and C_1 are square of the same size. These are compatible partitionings and block multiplication gives

$$AA_1 = \begin{bmatrix} B & X \\ 0 & C \end{bmatrix}\begin{bmatrix} B_1 & X_1 \\ 0 & C_1 \end{bmatrix} = \begin{bmatrix} BB_1 & BX_1 + XC_1 \\ 0 & CC_1 \end{bmatrix}$$

Block multiplication is useful in computing products of matrices in a computer with limited memory capacity. The matrices are partitioned into blocks in such a way that each product of blocks can be handled. Then the blocks are stored in auxiliary memory (on tape, for example), and the products are computed one by one.

Exercises 2.2

1. Compute the following matrix products.

(a) $\begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$ ♦(b) $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 1 & 9 & 7 \\ -1 & 0 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 5 & 0 & -7 \\ 1 & 5 & 9 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$ ♦(d) $\begin{bmatrix} 1 & 3 & -3 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -2 & 1 \\ 0 & 6 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 5 & -7 \\ 9 & 7 \end{bmatrix}$ ♦(f) $\begin{bmatrix} 1 & -1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -8 \end{bmatrix}$

(g) $\begin{bmatrix} 2 \\ 1 \\ -7 \end{bmatrix}\begin{bmatrix} 1 & -1 & 3 \end{bmatrix}$ ♦(h) $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

(i) $\begin{bmatrix} 2 & 3 & 1 \\ 5 & 7 & 4 \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ ♦(j) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} a' & 0 & 0 \\ 0 & b' & 0 \\ 0 & 0 & c' \end{bmatrix}$

2. In each of the following cases, find all possible products A^2, AB, AC, and so on.

(a) $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ \frac{1}{2} & 3 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 0 \\ 2 & 5 \\ 0 & 3 \end{bmatrix}$

♦(b) $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 6 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 1 & 2 \end{bmatrix}$

3. Find a, b, a_1, and b_1 if:

(a) $\begin{bmatrix} a & b \\ a_1 & b_1 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$

♦(b) $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ a_1 & b_1 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ -1 & 4 \end{bmatrix}$

4. Verify that $A^2 - A - 6I = 0$ if:

(a) $A = \begin{bmatrix} 3 & -1 \\ 0 & -2 \end{bmatrix}$ ♦(b) $A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$

5. Given $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 5 & 8 \end{bmatrix}$,

and $D = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 5 \end{bmatrix}$, verify the following facts from Theorem 1.

(a) $A(B - D) = AB - AD$ ♦(b) $A(BC) = (AB)C$

(c) $(CD)^T = D^T C^T$

6. Let A be a 2×2 matrix.

(a) If A commutes with $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, show that $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ for some a and b.

♦(b) If A commutes with $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, show that $A = \begin{bmatrix} a & 0 \\ c & a \end{bmatrix}$ for some a and c.

(c) Show that A commutes with *every* 2×2 matrix if and only if $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ for some a.

7. Write each of the following systems of linear equations in matrix form.

(a)
$$\begin{aligned} 3x_1 + 2x_2 - x_3 + x_4 &= 1 \\ x_1 - x_2 + 3x_4 &= 0 \\ 2x_1 - x_2 - x_3 &= 5 \end{aligned}$$

♦(b)
$$\begin{aligned} -x_1 + 2x_2 - x_3 + x_4 &= 6 \\ 2x_1 + x_2 - x_3 + 2x_4 &= 1 \\ 3x_1 - 2x_2 + x_4 &= 0 \end{aligned}$$

8. In each case, express every solution of the system as a sum of a specific solution plus a solution of the associated homogeneous system.

(a)
$$\begin{aligned} x + y + z &= 2 \\ 2x + y &= 3 \\ x - y - 3z &= 0 \end{aligned}$$

♦(b)
$$\begin{aligned} x - y - 4z &= -4 \\ x + 2y + 5z &= 2 \\ x + y + 2z &= 0 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + x_2 - x_3 - 5x_5 &= 2 \\ x_2 + x_3 - 4x_5 &= -1 \\ x_2 + x_3 + x_4 - x_5 &= -1 \\ 2x_1 - 4x_3 + x_4 + x_5 &= 6 \end{aligned}$$

♦(d)
$$\begin{aligned} 2x_1 + x_2 - x_3 - x_4 &= -1 \\ 3x_1 + x_2 + x_3 - 2x_4 &= -2 \\ -x_1 - x_2 + 2x_3 + x_4 &= 2 \\ -2x_1 - x_2 + 2x_4 &= 3 \end{aligned}$$

9. If X_0 and X_1 are solutions to the homogeneous system of equations $AX = 0$, show that $sX_0 + tX_1$ is also a solution for any scalars s and t (called a **linear combination** of X_0 and X_1).

10. In each of the following, find the basic solutions, and write the general solution as a linear combination of the basic solutions.

(a)
$$\begin{aligned} x_1 + 2x_2 - x_3 + 2x_4 + x_5 &= 0 \\ x_1 + 2x_2 + 2x_3 + x_5 &= 0 \\ 2x_1 + 4x_2 - 2x_3 + 3x_4 + x_5 &= 0 \end{aligned}$$

♦(b)
$$\begin{aligned} x_1 + x_2 - 2x_3 + 3x_4 + 2x_5 &= 0 \\ 2x_1 - x_2 + 3x_3 + 4x_4 + x_5 &= 0 \\ -x_1 - 2x_2 + 3x_3 + x_4 &= 0 \\ 3x_1 + x_3 + 7x_4 + 2x_5 &= 0 \end{aligned}$$

11. Assume that $A\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = 0 = A\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$, and that $AX = B$ has a solution $X_0 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$. Find a two-parameter family of solutions to $AX = B$.

12. (a) If A^2 can be formed, what can be said about the size of A?

♦(b) If AB and BA can both be formed, describe the sizes of A and B.

(c) If ABC can be formed, A is 3×3, and C is 5×5, what size is B?

13. (a) Find two 2×2 matrices A such that $A^2 = 0$.

♦(b) Find three 2×2 matrices A such that (i) $A^2 = I$; (ii) $A^2 = A$.

(c) Find 2×2 matrices A and B such that $AB = 0$ but $BA \neq 0$.

14. Write $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, and let A be $3 \times n$ and B be $m \times 3$.

(a) Describe PA in terms of the rows of A.

♦(b) Describe BP in terms of the columns of B.

15. Let A, B, and C be as in Exercise 5. Find the (3, 1)-entry of CAB using exactly six numerical multiplications.

16. (a) Compute AB, using the indicated block partitioning.

$$A = \left[\begin{array}{cc|cc} 2 & -1 & 3 & 1 \\ 1 & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad B = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 5 & 1 \\ 1 & -1 & 0 \end{array}\right]$$

♦(b) Partition A and B in part (a) differently and compute AB again.

(c) Find A^2 using the partitioning in part (a) and then again using a different partitioning.

17. In each case compute all powers of A using the block decomposition indicated.

(a) $A = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 1 & 1 & -1 \\ 1 & -1 & 1 \end{array}\right]$ ♦(b) $A = \left[\begin{array}{cc|cc} 1 & -1 & 2 & -1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$

18. Compute the following using block multiplication (all blocks $k \times k$).

(a) $\begin{bmatrix} I & X \\ -Y & I \end{bmatrix}\begin{bmatrix} I & 0 \\ Y & I \end{bmatrix}$ ♦(b) $\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}$

(c) $[I \;\; X][I \;\; X]^T$ ♦(d) $[I \;\; X^T][-X \;\; I]^T$

(e) $\begin{bmatrix} I & X \\ 0 & -I \end{bmatrix}^n$, any $n \geq 1$ ♦(f) $\begin{bmatrix} 0 & X \\ I & 0 \end{bmatrix}^n$, any $n \geq 1$

19. (a) If A has a row of zeros, sh ow that the same is true of AB for any B.

(b) If B has a column of zeros, show that the same is true of AB for any A.

20. Let A denote an $m \times n$ matrix.

(a) If $AX = 0$ for every $n \times 1$ matrix X, show that $A = 0$.

♦(b) If $YA = 0$ for every $1 \times m$ matrix Y, show that $A = 0$.

21. (a) If $U = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$, and $AU = 0$, show that $A = 0$.

(b) Let U be such that $AU = 0$ implies that $A = 0$. If $AU = BU$, show that $A = B$.

22. Simplify the following expressions where A, B, and C represent matrices.

(a) $A(3B - C) + (A - 2B)C + 2B(C + 2A)$

♦(b) $A(B + C - D) + B(C - A + D) - (A + B)C + (A - B)D$

(c) $AB(BC - CB) + (CA - AB)BC + CA(A - B)C$

♦(d) $(A - B)(C - A) + (C - B)(A - C) + (C - A)^2$

23. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a \neq 0$, show that A factors in the form $A = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}\begin{bmatrix} y & z \\ 0 & w \end{bmatrix}$.

24. If A and B commute with C, show that the same is true of:

(a) $A + B$ ♦(b) kA, k any scalar

25. If A is any matrix, show that AA^T and A^TA are symmetric.

♦26. If A and B are symmetric, show that AB is symmetric if and only if $AB = BA$.

27. If A is a 2×2 matrix, show that $A^TA = AA^T$ if and only if A is symmetric or $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ for some a and b.

28. (a) Find all symmetric 2×2 matrices A such that $A^2 = 0$.

♦(b) Repeat (a) if A is 3×3.

(c) Repeat (a) if A is $n \times n$.

29. Show that there exist no 2×2 matrices A and B such that $AB - BA = I$. [*Hint:* Examine the (1, 1)- and (2, 2)-entries.]

♦30. Let B be an $n \times n$ matrix. Suppose $AB = 0$ for some nonzero $m \times n$ matrix A. Show that no $n \times n$ matrix C exists such that $BC = I$.

31. (a) If A and B are 2×2 matrices whose rows sum to 1, show that the rows of AB also sum to 1.

♦(b) Repeat part (a) for the case where A and B are $n \times n$.

32. Let A and B be $n \times n$ matrices for which the systems of equations $AX = 0$ and $BX = 0$ each have only the trivial solution $X = 0$. Show that the system $(AB)X = 0$ has only the trivial solution.

33. The **trace** of a square matrix A, denoted tr A, is the sum of the elements on the main diagonal of A. Show that, if A and B are $n \times n$ matrices:

(a) $\text{tr}(A + B) = \text{tr } A + \text{tr } B$.

♦(b) $\text{tr}(kA) = k\, \text{tr}(A)$ for any number k.

(c) $\text{tr}(A^T) = \text{tr}(A)$.

(d) $\text{tr}(AB) = \text{tr}(BA)$.

♦(e) $\text{tr}(AA^T)$ is the sum of the squares of all entries of A.

34. Show that $AB - BA = I$ is impossible. [*Hint:* See the preceding exercise.]

35. A square matrix P is called an **idempotent** if $P^2 = P$. Show that:
    (a) 0 and I are idempotents.
    (b) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, and $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ are idempotents.
    (c) If P is an idempotent, so is $I - P$, and $P(I - P) = 0$.
    (d) If P is an idempotent, so is P^T.
    ♦(e) If P is an idempotent, so is $Q = P + AP - PAP$ for any square matrix A (of the same size as P).
    (f) If A is $n \times m$ and B is $m \times n$, and if $AB = I_n$, then BA is an idempotent.

36. Let A and B be $n \times n$ **diagonal matrices** (all entries off the main diagonal are zero).
    (a) Show that AB is diagonal and $AB = BA$.
    (b) Formulate a rule for calculating XA if X is $m \times n$.
    (c) Formulate a rule for calculating AY if Y is $n \times k$.

37. If A and B are $n \times n$ matrices, show that:
    (a) $AB = BA$ if and only if $(A + B)^2 = A^2 + 2AB + B^2$.
    ♦(b) $AB = BA$ if and only if $(A + B)(A - B) = (A - B)(A + B)$.
    (c) $AB = BA$ if and only if $A^TB^T = B^TA^T$.

38. Prove the following parts of Theorem 1.
    (a) Part 1
    ♦(b) Part 2
    (c) Part 4
    (d) Part 5

Section 2.3 Matrix Inverses

Three basic operations on matrices, addition, multiplication, and subtraction, are analogues for matrices of the same operations for numbers. In this section we introduce the matrix analogue of numerical division.

To begin, consider how a numerical equation

$$ax = b$$

is solved when a and b are known numbers. If $a = 0$, there is no solution (unless $b = 0$). But if $a \neq 0$, we can multiply both sides by the inverse a^{-1} to obtain the solution $x = a^{-1}b$. This multiplication by a^{-1} is commonly called dividing by a, and the property of a^{-1} that makes this work is that $a^{-1}a = 1$. Moreover, we saw in Section 2.2 that the role that 1 plays in arithmetic is played in matrix algebra by the identity matrix I.

This suggests the following definition. If A is a square matrix, a matrix B is called an **inverse** of A if and only if

$$AB = I \quad \text{and} \quad BA = I$$

A matrix A that has an inverse is called an **invertible matrix.**[4]

Example 1

Show that $B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ is an inverse of $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

Solution

Compute AB and BA.

$$AB = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad BA = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence $AB = I = BA$, so B is indeed an inverse of A.

[4] Only square matrices have inverses. Even though it is plausible that nonsquare matrices A and B exist such that $AB = I_m$ and $BA = I_n$, where A is $m \times n$ and B is $n \times m$, we claim that this forces $n = m$. Indeed, if $m > n$ there exists a nonzero column X such that $AX = 0$ (by Theorem 1 §1.3), so $X = I_nX = (BA)X = B(AX) = B(0) = 0$, a contradiction. Hence $m \leq n$. Similarly, the condition $AB = I_m$ implies that $n \leq m$.

Example 2

Show that $A = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}$ has no inverse.

Solution

Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ denote an arbitrary 2×2 matrix. Then

$$AB = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a+3c & b+3d \end{bmatrix}$$

so AB has a row of zeros. Hence AB cannot equal I for any B.

Example 2 shows that *it is possible for a nonzero matrix to have no inverse*. But if a matrix *does* have an inverse, it has only one.

Theorem 1

If B and C are both inverses of A, then $B = C$.

Proof

Since B and C are both inverses of A, we have $CA = I = AB$. Hence $B = IB = (CA)B = C(AB) = CI = C$.

If A is an invertible matrix, the (unique) inverse of A is denoted as A^{-1}. Hence A^{-1} (when it exists) is a square matrix of the same size as A with the property that

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I$$

These equations characterize A^{-1} in the following sense: If somehow a matrix B can be found such that $AB = I = BA$, then A is invertible and B is the inverse of A; in symbols, $B = A^{-1}$. This gives us a way of verifying that the inverse of a matrix exists. Examples 3 and 4 offer illustrations.

Example 3

If $A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$, show that $A^3 = I$ and so find A^{-1}.

Solution

We have $A^2 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, and so

$$A^3 = A^2A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence $A^3 = I$, as asserted. This can be written as $A^2A = I = AA^2$, so it shows that A^2 is the inverse of A. That is, $A^{-1} = A^2 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$.

The following example gives a useful formula for the inverse of a 2×2 matrix.

Example 4

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc \neq 0$, show that $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Solution

We verify that $AA^{-1} = I$ and leave $A^{-1}A = I$ to the reader.

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) &= \frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = I \end{aligned}$$

Inverses and Linear Systems

Matrix inverses can be used to solve certain systems of linear equations. Recall (Example 7 §2.2) that a *system* of linear equations can be written as a *single* matrix equation

$$AX = B$$

where A and B are known matrices and X is to be determined. If A is invertible, we multiply each side of the equation on the left by A^{-1} to get

$$\begin{aligned} A^{-1}AX &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

This gives the solution to the system of equations (the reader should verify that $X = A^{-1}B$ really does satisfy $AX = B$). Furthermore, the argument shows that if X is *any* solution, then necessarily $X = A^{-1}B$, so the solution is unique. Of course the technique works only when the coefficient matrix A has an inverse. This proves Theorem 2.

Theorem 2

Suppose a system of n equations in n variables is written in matrix form as

$$AX = B$$

If the $n \times n$ coefficient matrix A is invertible, the system has the unique solution

$$X = A^{-1}B$$

Example 5

If $A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, show that $A^{-1} = \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 3 \\ -1 & -2 & 5 \end{bmatrix}$, and use it to solve the following system of linear equations.

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 3 \\ 2x_1 + x_2 + x_3 &= 0 \\ x_1 + x_3 &= -2 \end{aligned}$$

Solution

The verification that $AA^{-1} = I$ and $A^{-1}A = I$ is left to the reader. The matrix form of the system of equations is $AX = B$, where A is as before and

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

Theorem 2 gives the solution

$$X = A^{-1}B = \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 3 \\ -1 & -2 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 11 \\ -9 \\ -13 \end{bmatrix}$$

Thus, $x_1 = 11$, $x_2 = -9$, and $x_3 = -13$.

An Inversion Method

Given a particular $n \times n$ matrix A, it is desirable to have an efficient technique to determine whether A has an inverse and, if so, to find that inverse. For simplicity, we shall derive the technique for 2×2 matrices; the $n \times n$ case is entirely analogous.

Given the invertible 2×2 matrix A, we determine A^{-1} from the equation $AA^{-1} = I$. Write

$$A^{-1} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

where x_1, y_1, x_2, and y_2 are to be determined. Equating columns in the equation $AA^{-1} = I$ gives

$$A\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad A\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

These are systems of linear equations, each with A as coefficient matrix. Since A is invertible, each system has a unique solution by Theorem 2. But this means that the reduced row-echelon form R of A cannot have a row of zeros, and so is the identity matrix (R is square). Hence, there is a sequence of elementary row operations carrying A to the 2×2 identity matrix I. This sequence carries the augmented matrices of both systems to reduced row-echelon form and so solves the systems:

$$\left[\begin{array}{c|c} A & \begin{matrix} 1 \\ 0 \end{matrix} \end{array}\right] \to \left[\begin{array}{c|c} I & \begin{matrix} x_1 \\ y_1 \end{matrix} \end{array}\right] \qquad \left[\begin{array}{c|c} A & \begin{matrix} 0 \\ 1 \end{matrix} \end{array}\right] \to \left[\begin{array}{c|c} I & \begin{matrix} x_2 \\ y_2 \end{matrix} \end{array}\right]$$

Hence, we can do *both* calculations simultaneously.

$$\left[\begin{array}{c|cc} A & 1 & 0 \\ & 0 & 1 \end{array}\right] \to \left[\begin{array}{c|cc} I & x_1 & x_2 \\ & y_1 & y_2 \end{array}\right]$$

This can be written more compactly as follows:

$$[A \;\; I] \to [I \;\; A^{-1}]$$

In other words, the sequence of row operations that carries A to I also carries I to A^{-1}. This is the desired algorithm.

Matrix Inversion Algorithm

If A is a (square) invertible matrix, there exists a sequence of elementary row operations that carry A to the identity matrix I of the same size, written $A \to I$. This same series of row operations carries I to A^{-1}; that is, $I \to A^{-1}$. The algorithm can be summarized as follows:

$$[A \quad I] \to [I \quad A^{-1}]$$

where the row operations on A and I are carried out simultaneously.

Example 6

Use the inversion algorithm to find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix}$$

Solution

Apply elementary row operations to the double matrix

$$[A \quad I] = \left[\begin{array}{rrr|rrr} 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

so as to carry A to I. First interchange rows 1 and 2.

$$\left[\begin{array}{rrr|rrr} 1 & 4 & -1 & 0 & 1 & 0 \\ 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

Next subtract 2 times row 1 from row 2, and subtract row 1 from row 3.

$$\left[\begin{array}{rrr|rrr} 1 & 4 & -1 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{array}\right]$$

Continue to reduced row-echelon form.

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{-3}{2} & \frac{-3}{2} & \frac{11}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{-3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \end{array}\right]$$

Hence $A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -3 & 11 \\ 1 & 1 & -3 \\ 1 & -1 & -1 \end{bmatrix}$, as is readily verified.

Given any $n \times n$ matrix A, Theorem 1 §1.2 shows that A can be carried by elementary row operations to a matrix R in reduced row-echelon form. If $R = I$, the matrix A is invertible (this will be proved in the next section), so the algorithm produces A^{-1}. If $R \neq I$, then R has a row of zeros (it is square), so no system of linear equations $AX = B$ can have a unique solution. But then A is not invertible by Theorem 2. Hence, the algorithm is effective in the sense conveyed in Theorem 3.

Theorem 3

If A is an $n \times n$ matrix, either A can be reduced to I by elementary row operations or it cannot. In the first case, the algorithm produces A^{-1}; in the second case, A^{-1} does not exist.

Properties of Inverses

Sometimes the inverse of a matrix is given by a formula. Example 4 is one illustration, Examples 7 and 8 provide two more. Given a square matrix A, recall that if a matrix B can be found such that $AB = I = BA$, then A is invertible and $A^{-1} = B$.

Example 7

If A is an invertible matrix, show that the transpose A^T is also invertible. Show further that the inverse of A^T is just the transpose of A^{-1}; in symbols, $(A^T)^{-1} = (A^{-1})^T$.

Solution

A^{-1} exists (by assumption). Its transpose $(A^{-1})^T$ is the candidate proposed for the inverse of A^T. We test it as follows:

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$
$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

Hence $(A^{-1})^T$ is indeed the inverse of A^T; that is, $(A^T)^{-1} = (A^{-1})^T$.

Example 8

If A and B are invertible $n \times n$ matrices, show that their product AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Solution

We are given a candidate for the inverse of AB, namely $B^{-1}A^{-1}$. We test it as follows:

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$
$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

Hence $B^{-1}A^{-1}$ is the inverse of AB; in symbols, $(AB)^{-1} = B^{-1}A^{-1}$.

We now collect several basic properties of matrix inverses for reference.

Theorem 4

All the following matrices are square matrices of the same size.

1. I is invertible and $I^{-1} = I$.
2. If A is invertible, so is A^{-1}, and $(A^{-1})^{-1} = A$.
3. If A and B are invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.
4. If $A_1, A_2, \dots, A_k$ are all invertible, so is their product $A_1A_2 \cdots A_k$, and $(A_1A_2 \cdots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1}A_1^{-1}$.
5. If A is invertible, so is A^k for $k \geq 1$, and $(A^k)^{-1} = (A^{-1})^k$.
6. If A is invertible and $a \neq 0$ is a number, then aA is invertible and $(aA)^{-1} = \frac{1}{a}A^{-1}$.
7. If A is invertible, so is its transpose A^T, and $(A^T)^{-1} = (A^{-1})^T$.

Proof

1. This is an immediate consequence of the formula $I^2 = I$.
2. The equations $AA^{-1} = I = A^{-1}A$ show that A is the inverse of A^{-1}; in symbols, $(A^{-1})^{-1} = A$.
3. This is Example 8.
4. Use induction on k. If $k = 1$, there is nothing to prove because the conclusion reads $(A_1)^{-1} = A_1^{-1}$. If $k = 2$, the result is just property 3. If $k > 2$, assume inductively that $(A_1A_2 \cdots A_{k-1})^{-1} = A_{k-1}^{-1} \cdots A_2^{-1}A_1^{-1}$. We apply this fact together with property 3 as follows:

$$\begin{aligned}[A_1A_2 \cdots A_{k-1}A_k]^{-1} &= [(A_1A_2 \cdots A_{k-1})A_k]^{-1} \\ &= A_k^{-1}(A_1A_2 \cdots A_{k-1})^{-1} \\ &= A_k^{-1}(A_{k-1}^{-1} \cdots A_2^{-1}A_1^{-1})\end{aligned}$$

   Here property 3 is applied to get the second equality. This is the conclusion for k matrices, so the proof by induction is complete.
5. This is property 4 with $A_1 = A_2 = \cdots = A_k = A$.
6. This is left as Exercise 28.
7. This is Example 7.

Part 7 of Theorem 4 together with the fact that $(A^T)^T = A$ give

Corollary

A square matrix A is invertible if and only if A^T is invertible.

Example 9

Find A if $(A^T - 2I)^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$.

Solution

By Theorem 4(2) and Example 4

$$(A^T - 2I) = [(A^T - 2I)^{-1}]^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$$

Hence $A^T = 2I + \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$, so $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$.

The reversal of the order of the inverses in properties 3 and 4 of Theorem 4 is a consequence of the fact that matrix multiplication is not commutative. Another manifestation of this comes when matrix equations are dealt with. If a matrix equation $B = C$ is given, it can be *left-multiplied* by a matrix A to yield $AB = AC$. Similarly, *right-multiplication* gives $BA = CA$. However, we cannot mix the two: If $B = C$, it need *not* be the case that $AB = CA$.

We conclude this section with an important theorem that collects a number of conditions all equivalent[5] to invertibility. It will be referred to frequently below.

[5] If p and q are statements, we say that p **implies** q (written $p \Rightarrow q$) if q is true whenever p is true. The statements are called **equivalent** if both $p \Rightarrow q$ and $q \Rightarrow p$ (written $p \Leftrightarrow q$, spoken "p if and only if q").

Theorem 5

The following conditions are equivalent for an $n \times n$ matrix A:

1. A is invertible.
2. The homogeneous system $AX = 0$ has only the trivial solution $X = 0$.
3. A can be carried to the identity matrix I_n by elementary row operations.
4. The system $AX = B$ has at least one solution X for every choice of column B.
5. There exists an $n \times n$ matrix C such that $AC = I_n$.

Proof

We show that each of these conditions implies the next, and that (5) implies (1).

(1) $\Rightarrow$ (2). If A^{-1} exists then $AX = 0$ gives $X = I_nX = A^{-1}AX = A^{-1}0 = 0$. This is (2).

(2) $\Rightarrow$ (3). Assume that (2) is true. Certainly $A \to R$ by row operations where R is a reduced, row-echelon matrix; we show that $R = I_n$. Suppose on the contrary that $R \neq I_n$. Then R has a row of zeros (being square), and we consider the augmented matrix $[A \;\; 0]$ of the system $AX = 0$. Then $[A \;\; 0] \to [R \;\; 0]$ is the row-echelon form and $[R \;\; 0]$ also has a row of zeros. Since A is square, this means that there is at least one non-leading variable, and hence at least one parameter. Thus $AX = 0$ has infinitely many solutions, contrary to (2). So $R = I_n$ after all.

(3) $\Rightarrow$ (4). Consider the augmented matrix $[A \;\; B]$ of the system. Using (3) let $A \to I$ by a sequence of row operations. Then these same operations carry $[A \;\; B] \to [I \;\; C]$ for some column C. Hence the system $AX = B$ has a solution (in fact unique) by Gaussian elimination.

(4) $\Rightarrow$ (5). Write $I_n = [E_1 \;\; E_2 \cdots E_n]$ where $E_1, E_2, \ldots, E_n$ are the columns of I_n. For each $j = 1, 2, \ldots, n$, the system $AX = E_j$ has a solution C_j by (4), so $AC_j = E_j$. Now let $C = [C_1 \;\; C_2 \cdots C_n]$ be the $n \times n$ matrix with these matrices C_j as its columns. Then the definition of matrix multiplication gives (5):

$$AC = A[C_1 \; C_2 \cdots C_n] = [AC_1 \; AC_2 \cdots AC_n] = [E_1 \; E_2 \cdots E_n] = I_n$$

(5) $\Rightarrow$ (1). Assume that (5) is true so that $AC = I_n$ for some matrix C. Then $CX = 0$ implies $X = 0$ (because $X = I_nX = ACX = A0 = 0$). Thus condition (2) holds for the matrix C rather than A. Hence the argument above that (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) (with A replaced by C) shows that a matrix C' exists such that $CC' = I$. But then

$$A = AI_n = A(CC') = (AC)C' = IC' = C'$$

Thus $CA = CC' = I$ which, together with $AC = I$, shows that C is the inverse of A. This proves (1).

The proof of (5) $\Rightarrow$ (1) in Theorem 5 shows that if $AC = I$ for square matrices, then necessarily $CA = I$, and hence that C and A are inverses of each other. We record this important fact for reference.

Corollary

If A and C are square matrices such that $AC = I$, then also $CA = I$. In particular, both A and C are invertible, $C = A^{-1}$, and $A = C^{-1}$.

Observe that the Corollary is false if A and C are not square matrices. For example, we have

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = I_2 \quad \text{but} \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \neq I_3$$

In fact, it is verified in the footnote on page 50 that if $AB = I_m$ and $BA = I_n$, where A is $m \times n$ and B is $n \times m$, then $m = n$ and A and B are (square) inverses of each other.

Example 10

Show that $A = \begin{bmatrix} 6 & 8 \\ 15 & 20 \end{bmatrix}$ has no inverse.

Solution

Observe that $AX = 0$ where $X = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$. Hence A has no inverse by Part (2) of Theorem 5.

Note that we do not need Theorem 5 for this: If A^{-1} exists then left-multiplying $AX = 0$ by A^{-1} gives $A^{-1}AX = A^{-1}0$, that is $IX = 0$. This means that $X = 0$, which is not the case. So A^{-1} does not exist.

Exercises 2.3

1. In each case, show that the matrices are inverses of each other.

(a) $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 0 \\ 1 & -4 \end{bmatrix}, \frac{1}{12}\begin{bmatrix} 4 & 0 \\ 1 & -3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 7 & 2 & -6 \\ -3 & -1 & 3 \\ 2 & 1 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$

2. Find the inverse of each of the following matrices.

(a) $\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$ ♦(b) $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 0 \\ -1 & -1 & 0 \end{bmatrix}$ ♦(d) $\begin{bmatrix} 1 & -1 & 2 \\ -5 & 7 & -11 \\ -2 & 3 & -5 \end{bmatrix}$

(e) $\begin{bmatrix} 3 & 5 & 0 \\ 3 & 7 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ ♦(f) $\begin{bmatrix} 3 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & 5 & -1 \end{bmatrix}$

(g) $\begin{bmatrix} 2 & 4 & 1 \\ 3 & 3 & 2 \\ 4 & 1 & 4 \end{bmatrix}$ ♦(h) $\begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix}$

(i) $\begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$ ♦(j) $\begin{bmatrix} -1 & 4 & 5 & 2 \\ 0 & 0 & 0 & -1 \\ 1 & -2 & -2 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix}$

(k) $\begin{bmatrix} 1 & 0 & 7 & 5 \\ 0 & 1 & 3 & 6 \\ 1 & -1 & 5 & 2 \\ 1 & -1 & 5 & 1 \end{bmatrix}$ ♦(l) $\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

3. In each case, solve the systems of equations by finding the inverse of the coefficient matrix.

(a) $\begin{aligned} 3x - y &= 5 \\ 2x + 3y &= 1 \end{aligned}$ ♦(b) $\begin{aligned} 2x - 3y &= 0 \\ x - 4y &= 1 \end{aligned}$

(c) $\begin{aligned} x + y + 2z &= 5 \\ x + y + z &= 0 \\ x + 2y + 4z &= -2 \end{aligned}$

♦(d) $\begin{aligned} x + 4y + 2z &= 1 \\ 2x + 3y + 3z &= -1 \\ 4x + y + 4z &= 0 \end{aligned}$

(e) $\begin{aligned} x + y - w &= 1 \\ -x + y - z &= -1 \\ y + z + w &= 0 \\ x - z + w &= 1 \end{aligned}$

♦(f) $\begin{aligned} x + y + z + w &= 1 \\ x + y &= 0 \\ y + w &= -1 \\ x + w &= 2 \end{aligned}$

4. Given $A^{-1} = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \\ -1 & 1 & 0 \end{bmatrix}$:

(a) Solve the system of equations $AX = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$.

♦(b) Find a matrix B such that $AB = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

(c) Find a matrix C such that $CA = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \end{bmatrix}$.

5. Find A when

(a) $(3A)^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ ♦(b) $(2A)^T = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}^{-1}$

(c) $(I + 3A)^{-1} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$

♦(d) $(I - 2A^T)^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

(e) $\left(A\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$

♦(f) $\left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}A\right)^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$

(g) $(A^T - 2I)^{-1} = 2\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

♦(h) $(A^{-1} - 2I)^T = -2\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

6. Find A when:

(a) $A^{-1} = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 1 \\ 0 & 2 & -2 \end{bmatrix}$ ♦(b) $A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

7. Given $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 4 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and $\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 0 \\ -1 & 1 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, express the variables x_1, x_2, and x_3 in terms of z_1, z_2, and z_3.

8. (a) In the system $\begin{matrix} 3x + 4y = 7 \\ 4x + 5y = 1 \end{matrix}$, substitute the new variables x' and y' given by $\begin{matrix} x = -5x' + 4y' \\ y = 4x' - 3y' \end{matrix}$. Then find x and y.

♦(b) Explain part (a) by writing the equations as $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix} = B\begin{bmatrix} x' \\ y' \end{bmatrix}$. What is the relationship between A and B? Generalize.

9. In each case either prove the assertion or give an example showing that it is false.

(a) If $A \neq 0$ is a square matrix, then A is invertible.

♦(b) If A and B are both invertible, then $A + B$ is invertible.

(c) If A and B are both invertible, then $(A^{-1}B)^T$ is invertible.

♦(d) If $A^4 = 3I$, then A is invertible.

(e) If $A^2 = A$ and $A \neq 0$, then A is invertible.

♦(f) If $AB = B$ for some $B \neq 0$, then A is invertible.

(g) If A is invertible and skew symmetric ($A^T = -A$), the same is true of A^{-1}.

♦10. If A, B, and C are square matrices and $AB = I = CA$, show that A is invertible and $B = C = A^{-1}$.

11. Suppose $CA = I_m$, where C is $m \times n$ and A is $n \times m$. Consider the system $AX = B$ of n equations in m variables.

(a) Show that this system has a unique solution CB if it is consistent.

♦(b) If $C = \begin{bmatrix} 0 & -5 & 1 \\ 3 & 0 & -1 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \\ 6 & -10 \end{bmatrix}$, find X (if it exists) when (i) $B = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$; and (ii) $B = \begin{bmatrix} 7 \\ 4 \\ 22 \end{bmatrix}$.

12. Verify that $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ satisfies $A^2 - 3A + 2I = 0$, and use this fact to show that $A^{-1} = \frac{1}{2}(3I - A)$.

13. Let $Q = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}$. Compute QQ^T and so find Q^{-1}.

14. Let $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Show that each of U, $-U$, and $-I_2$ is its own inverse and that the product of any two of these is the third.

15. Consider $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{bmatrix}$. Find the inverses by computing (a) A^6; ♦(b) B^4; and (c) C^3.

16. In each case, find A^{-1} in terms of c.

(a) $\begin{bmatrix} c & -1 \\ -1 & c \end{bmatrix}$ ♦(b) $\begin{bmatrix} 2 & -c \\ c & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & c & 0 \\ 2 & c & 1 \\ c & -1 & c \end{bmatrix}$ ♦(d) $\begin{bmatrix} 1 & 0 & 1 \\ c & 1 & c \\ 3 & c & 2 \end{bmatrix}$

17. If $c \neq 0$, find the inverse of $\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 2 \\ 0 & 2 & c \end{bmatrix}$ in terms of c.

♦18. Find the inverse of $\begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$ for any real number θ.

19. Show that A has no inverse when

(a) A has a row of zeros;

♦(b) A has a column of zeros;

(c) each row of A sums to 0;

♦(d) each column of A sums to 0.

20. Let A denote a square matrix.

(a) Let $YA = 0$ for some matrix $Y \neq 0$. Show that A has no inverse.

(b) Use part (a) to show that (i) $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$; and

♦(ii) $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ have no inverse.

[*Hint:* For part (ii) compare row 3 with the difference between row 1 and row 2.]

21. If A is invertible, show that

(a) $A^2 \neq 0$;

(b) $A^k \neq 0$ for all $k = 1, 2, \dots$;

(c) $AX = AY$ implies $X = Y$;

♦(d) $PA = QA$ implies $P = Q$.

22. Suppose $AB = 0$, where A and B are square matrices. Show that:

(a) If one of A and B has an inverse, the other is zero.

♦(b) It is impossible for both A and B to have inverses.

(c) $(BA)^2 = 0$.

23. (a) Show that $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is invertible if and only if $a \neq 0$ and $b \neq 0$. Describe the inverse.

(b) Show that a diagonal matrix is invertible if and only if all the main diagonal entries are nonzero. Describe the inverse.

(c) If A and B are square matrices, show that

(i) the block matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is invertible if and only if A and B are both invertible; and

(ii) $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix}$.

(d) Use part (c) to find the inverses of:

(i) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix}$ ♦(ii) $\begin{bmatrix} 3 & 1 & 0 \\ 5 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ ♦(iv) $\begin{bmatrix} 3 & 4 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

(e) Extend part (c) to **block diagonal matrices**—that is, matrices with square blocks down the main diagonal and zero blocks elsewhere.

24. (a) Show that $\begin{bmatrix} a & x \\ 0 & b \end{bmatrix}$ is invertible if and only if $a \neq 0$ and $b \neq 0$.

♦(b) If A and B are square and invertible, show that (i) the block matrix $\begin{bmatrix} A & X \\ 0 & B \end{bmatrix}$ is invertible for any X; and

(ii) $\begin{bmatrix} A & X \\ 0 & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}XB^{-1} \\ 0 & B^{-1} \end{bmatrix}$.

(c) Use part (b) to invert (i) $\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$;

and ♦(ii) $\begin{bmatrix} 3 & 1 & 3 & 0 \\ 2 & 1 & -1 & 1 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 3 & 1 \end{bmatrix}$.

25. If A and B are invertible symmetric matrices such that $AB = BA$, show that A^{-1}, AB, AB^{-1}, and $A^{-1}B^{-1}$ are also invertible and symmetric.

26. (a) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Verify that $AB = CA$, A is invertible, but $B \neq C$. (See Exercise 21.)

♦(b) Find 2×2 matrices P, Q, and R such that $PQ = PR$, P is not invertible, and $Q \neq R$. (See Exercise 21.)

27. Let A be an $n \times n$ matrix and let I be the $n \times n$ identity matrix.

(a) If $A^2 = 0$, verify that $(I - A)^{-1} = I + A$.

(b) If $A^3 = 0$, verify that $(I - A)^{-1} = I + A + A^2$.

(c) Using part (b), find the inverse of $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$.

♦(d) If $A^n = 0$, find the formula for $(I - A)^{-1}$.

28. Prove property 6 of Theorem 4; If A is invertible and $a \neq 0$, then aA is invertible and $(aA)^{-1} = \frac{1}{a}A^{-1}$.

29. Let A, B, and C denote $n \times n$ matrices. Show that:

(a) If A and AB are both invertible, B is invertible.

♦(b) If AB and BA are both invertible, A and B are both invertible. [*Hint:* See Exercise 10.]

(c) If A, C, and ABC are all invertible, B is invertible.

30. Let A and B denote invertible $n \times n$ matrices.

(a) If $A^{-1} = B^{-1}$, does it mean that $A = B$? Explain.

♦(b) Show that $A = B$ if and only if $A^{-1}B = I$.

31. Let A, B, and C be $n \times n$ matrices, with A and B invertible. Show that

♦(a) If A commutes with C, then A^{-1} commutes with C.

(b) If A commutes with B, then A^{-1} commutes with B^{-1}.

32. Let A and B be square matrices of the same size.

(a) Show that $(AB)^2 = A^2B^2$ if $AB = BA$.

♦(b) If A and B are invertible and $(AB)^2 = A^2B^2$, show that $AB = BA$.

(c) If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, show that $(AB)^2 = A^2B^2$ but $AB \neq BA$.

33. Show that the following are equivalent for $n \times n$ matrices A and B.

(i) A and B are both invertible.

♦(ii) AB is invertible.

34. Consider $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 5 \\ 1 & -7 & 13 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 0 & -3 \\ -2 & 5 & 17 \end{bmatrix}$.

(a) Show that A is not invertible by finding a nonzero 1×3 matrix Y such that $YA = 0$. [*Hint:* Row 3 of A equals 2(row 2) – 3(row 1).]

♦(b) Show that B is not invertible. [*Hint:* Column 3 = 3(column 2) – column 1.]

♦35. Show that a square matrix A is invertible if and only if it can be left-cancelled: $AB = AC$ implies $B = C$.

36. If $U^2 = I$, show that $I + U$ is not invertible unless $U = I$.

37. (a) If J is the 4×4 matrix with every entry 1, show that $I - \frac{1}{2}J$ is self-inverse and symmetric.

(b) If X is $n \times m$ and satisfies $X^TX = I_m$, show that $I_n - 2XX^T$ is self-inverse and symmetric.

38. An $n \times n$ matrix P is called an idempotent if $P^2 = P$. Show that:

(a) I is the only invertible idempotent.

♦(b) P is an idempotent if and only if $I - 2P$ is self-inverse.

(c) U is self-inverse if and only if $U = I - 2P$ for some idempotent P.

(d) $I - aP$ is invertible for any $a \neq 1$, and $(I - aP)^{-1} = I + \left(\frac{a}{1-a}\right)P$.

39. If $A^2 = kA$, where $k \neq 0$, show that A is invertible if and only if $A = kI$.

40. Let A and B denote $n \times n$ invertible matrices.

(a) Show that $A^{-1} + B^{-1} = A^{-1}(A + B)B^{-1}$.

♦(b) If $A + B$ is also invertible, show that $A^{-1} + B^{-1}$ is invertible and find a formula for $(A^{-1} + B^{-1})^{-1}$.

41. Let A and B be $n \times n$ matrices, and let I be the $n \times n$ identity matrix.

(a) Verify that $A(I + BA) = (I + AB)A$ and that $(I + BA)B = B(I + AB)$.

(b) If $I + AB$ is invertible, verify that $I + BA$ is also invertible and that $(I + BA)^{-1} = I - B(I + AB)^{-1}A$.

Section 2.4 Elementary Matrices

It is now evident that elementary row operations play a fundamental role in linear algebra by providing a general method for solving systems of linear equations. This leads to the matrix inversion algorithm. It turns out that these elementary row operations can be performed by left-multiplying by certain invertible matrices (called elementary matrices). This section is devoted to a discussion of this useful fact and some of its consequences.

Recall that the elementary row operations are of three types:

Type I: Interchange two rows.

Type II: Multiply a row by a nonzero number.

Type III: Add a multiple of a row to a different row.

An $n \times n$ matrix is called an **elementary matrix** if it is obtained from the $n \times n$ identity matrix by an elementary row operation. The elementary matrix so constructed is said to be of type I, II, or III when the corresponding row operation is of type I, II, or III.

Example 1

Verify that $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}$, and $E_3 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are elementary matrices.

Solution

E_1 is obtained from the 3×3 identity I_3 by interchanging the first two rows, so it is an elementary matrix of type I. Similarly, E_2 comes from multiplying the third row of I_3 by 9 and so is an elementary matrix of type II. Finally, E_3 is an elementary matrix of type III; it is obtained by adding 5 times the third row of I_3 to the first row.

Now consider the following three 2×2 elementary matrices E_1, E_2, and E_3 obtained by doing the indicated elementary row operations to I_2.

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{Interchange rows 1 and 2 of } I_2.$$

$$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \quad \text{Multiply row 2 of } I_2 \text{ by } k \neq 0.$$

$$E_3 = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \quad \text{Add } k \text{ times row 2 of } I_2 \text{ to row 1.}$$

If $A = \begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix}$ is any 2×3 matrix, we compute E_1A, E_2A, and E_3A:

$$E_1A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix} = \begin{bmatrix} p & q & r \\ a & b & c \end{bmatrix}$$

$$E_2A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix} = \begin{bmatrix} a & b & c \\ kp & kq & kr \end{bmatrix}$$

$$E_3A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix} = \begin{bmatrix} a+kp & b+kq & c+kr \\ p & q & r \end{bmatrix}$$

Observe that E_1A is the matrix resulting from interchanging rows 1 and 2 of A and that this row operation is the one that was used to produce E_1 from I_2. Similarly,

E_2A is obtained from A by the same row operation that produced E_2 from I_2 (multiplying row 2 by k). Finally, the same is true of E_3A: It is obtained from A by the same operation that produced E_3 from I_2 (adding k times row 2 to row 1). This phenomenon holds for arbitrary $m \times n$ matrices A.

Theorem 1

Let A denote any $m \times n$ matrix and let E be the $m \times m$ elementary matrix obtained by performing some elementary row operation on the $m \times m$ identity matrix I. If the same elementary row operation is performed on A, the resulting matrix is EA.

Proof

We prove it only for E of type III (types I and II are left as Exercise 14). If E is obtained by adding k times row p of I_m to row q, we must show that EA is obtained from A in the same way. Let $R_1, R_2, \dots, R_m$ and $K_1, K_2, \dots, K_m$ denote the rows of E and I, respectively. Then row i of $EA = R_iA = K_iA$ = row i of A if $i \neq q$. However, if $i = q$: row q of $EA = R_qA = (K_q + kK_p)A$ $= K_qA + kK_pA$. This is row q of A plus k times row p of A, as required.

Example 2

Given $A = \begin{bmatrix} 4 & 1 & 2 & 1 \\ 3 & 0 & 1 & 6 \\ 5 & 7 & 9 & 8 \end{bmatrix}$, find an elementary matrix E such that EA is the result of subtracting 7 times row 1 from row 3.

Solution

The elementary matrix is $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}$, obtained by doing the given row operation to I_3. The product

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 2 & 1 \\ 3 & 0 & 1 & 6 \\ 5 & 7 & 9 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 2 & 1 \\ 3 & 0 & 1 & 6 \\ -23 & 0 & -5 & 1 \end{bmatrix}$$

is indeed the result of applying the operation to A.

Given any elementary row operation, there is another row operation (called its **inverse**) that reverses the effect of the first operation. The inverses are described in the accompanying table.

Type	Operation	Inverse operation
I	Interchange rows p and q	Interchange rows p and q
II	Multiply row p by $c \neq 0$	Multiply row p by $\frac{1}{c}$
III	Add k times row p to row q ($p \neq q$)	Subtract k times row p from row q

Note that type I operations are self-inverse.

Theorem 2

Every elementary matrix E is invertible, and the inverse is an elementary matrix of the same type. More precisely:

E^{-1} is the elementary matrix obtained from I by the inverse of the row operation that produced E from I.

Proof

E is the result of applying a row operation ρ to I. Let E' denote the matrix obtained by applying the inverse operation ρ' to I. By Theorem 1, applying ρ to a matrix A produces EA; then applying ρ' to EA gives $E'(EA)$:

$$A \xrightarrow{\rho} EA \xrightarrow{\rho'} E'EA$$

But ρ' reverses the effect of ρ, so applying ρ followed by ρ' does not change A. Hence $E'EA = A$. In particular, taking $A = I$ gives $E'E = I$. A similar argument shows $EE' = I$, so $E^{-1} = E'$ as required.

Example 3

Write down the inverses of the elementary matrices E_1, E_2, and E_3 in Example 1.

Solution

The matrices are $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}$, and $E_3 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so they are of types I, II, and III, respectively. Hence Theorem 2 gives

$$E_1^{-1} = E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{9} \end{bmatrix} \qquad E_3^{-1} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now suppose a sequence of elementary row operations is performed on an $m \times n$ matrix A, and let $E_1, E_2, \dots, E_k$ denote the corresponding elementary matrices. Theorem 1 asserts that A is carried to E_1A under the first operation; in symbols, $A \to E_1A$. Then the second row operation is applied to E_1A (not to A) and the result is $E_2(E_1A)$, again by Theorem 1. Hence the reduction can be described as follows:

$$A \to E_1A \to E_2E_1A \to E_3E_2E_1A \to \cdots \to E_k \cdots E_2E_1A$$

In other words, the net effect of the *sequence* of elementary row operations is to left-multiply by the *product* $U = E_k \cdots E_2E_1$ of the corresponding elementary matrices (note the order). The result is

$$A \to UA \quad \text{where } U = E_k \cdots E_2E_1$$

Moreover, the matrix U can be easily constructed. Apply the same sequence of elementary operations to the $n \times n$ identity matrix I in place of A:

$$I \to UI = U$$

In other words, the sequence of elementary row operations that carries $A \to UA$ also carries $I \to U$. Hence it carries the double matrix $[A \;\; I]$ to $[UA \;\; U]$:

$$[A \quad I] \to [UA \quad U]$$

just as in the matrix inversion algorithm. This simple observation is surprisingly useful, and we record it as Theorem 3.

Theorem 3

Let A be an $m \times n$ matrix and assume that A can be carried to a matrix B by elementary row operations. Then:

1. $B = UA$ where U is an invertible $m \times m$ matrix.
2. $U = E_k E_{k-1} \cdots E_2 E_1$ where $E_1, E_2, \ldots, E_{k-1}, E_k$ are the elementary matrices corresponding (in order) to the elementary row operations that carry $A \to B$.
3. U can be constructed without finding the E_i by

$$[A \quad I] \to [UA \quad U]$$

In other words, the operations that carry $A \to UA$ also carry $I \to U$.

Proof

All that remains is to verify that U is invertible. Since U is a product of elementary matrices, this follows by Theorem 2.

Example 4

Find the reduced row-echelon form R of $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ and express it as $R = UA$, where U is invertible.

Solution

Use the usual row reduction $A \to R$ but carry out $I \to U$ simultaneously in the format $[A \;\; I] \to [R \;\; U]$.

$$\left[\begin{array}{ccc|cc} 2 & 3 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 0 & 1 \\ 2 & 3 & 1 & 1 & 0 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 0 & 1 \\ 0 & -1 & -1 & 1 & -2 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|cc} 1 & 0 & -1 & 2 & -3 \\ 0 & 1 & 1 & -1 & 2 \end{array}\right]$$

Hence, $R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$.

With Theorem 3 we can give an important characterization of invertible matrices in terms of elementary matrices.

Theorem 4

A square matrix A is invertible if and only if it is a product of elementary matrices.

Proof

If A is a product of elementary matrices, it is invertible by Theorem 2 because a product of invertible matrices is again invertible.

Conversely, assume that A is invertible. By Theorem 5 §2.3, A can be carried to the identity matrix I by elementary row operations. Hence, Theorem 3 (with $B = I$) implies that $I = UA$ where U is an invertible matrix that can be factored as a product $U = E_k E_{k-1} \cdots E_2 E_1$ of elementary matrices. But then

$$A = U^{-1} = (E_k E_{k-1} \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1}$$

and each E_i^{-1} is elementary by Theorem 2.

Example 5

Express $A = \begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix}$ as a product of elementary matrices.

Solution

We reduce A to I and write the elementary matrix at each stage.

$$\begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix} = A$$
$$\downarrow$$
$$\begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix} = E_1A \qquad \text{where } E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$\downarrow$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = E_2(E_1A) \qquad \text{where } E_2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$
$$\downarrow$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = E_3(E_2E_1A) \quad \text{where } E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

Hence $E_3E_2E_1A = I$ and so $A = (E_3E_2E_1)^{-1}$. This means that

$$A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

by Theorem 2. This is the desired factorization.

Exercises 2.4

1. For each of the following elementary matrices, describe the corresponding elementary row operation and write the inverse.

(a) $E = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ♦(b) $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(c) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ♦(d) $E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(e) $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ♦(f) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

2. In each case find an elementary matrix E such that $B = EA$.

(a) $A = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

♦(b) $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}$

♦(d) $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$

(e) $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$

♦(f) $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}$

3. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$.

(a) Find elementary matrices E_1 and E_2 such that $C = E_2E_1A$.

♦(b) Show that there is *no* elementary matrix E such that $C = EA$.

♦4. If E is elementary, show that A and EA differ in at most two rows.

5. (a) Is I an elementary matrix? Explain.

♦(b) Is 0 an elementary matrix? Explain.

6. In each case find an invertible matrix U such that $UA = R$ is in reduced row-echelon form, and express U as a product of elementary matrices.

(a) $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 0 \end{bmatrix}$

♦(b) $A = \begin{bmatrix} 1 & 2 & 1 \\ 5 & 12 & -1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & 1 & 1 & 2 \\ 1 & -3 & 3 & 2 \end{bmatrix}$

♦(d) $A = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 3 & -1 & 2 & 1 \\ 1 & -2 & 3 & 1 \end{bmatrix}$

7. In each case find an invertible matrix U such that $UA = B$, and express U as a product of elementary matrices.

(a) $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & -2 \\ 3 & 0 & 1 \end{bmatrix}$

♦(b) $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$

8. In each case factor A as a product of elementary matrices.

(a) $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ ♦(b) $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 6 \end{bmatrix}$ ♦(d) $A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ -2 & 2 & 15 \end{bmatrix}$

9. Let E be an elementary matrix.

(a) Show that E^T is also elementary of the same type.

(b) Show that $E^T = E$ if E is of type I or II.

♦10. Show that every matrix A can be factored as $A = UR$ where U is invertible and R is in reduced row-echelon form.

11. If $A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 2 \\ -5 & -3 \end{bmatrix}$, find an elementary matrix F such that $AF = B$. [*Hint:* See Exercise 9.]

12. Let $A = \frac{1}{7}\begin{bmatrix} 2 & 6 & -3 \\ 3 & 2 & 6 \end{bmatrix}$. Show that $AA^T = I_2$ but $A^TA \neq I_3$.

13. If $A = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ 1 & -1 \\ -1 & 0 \end{bmatrix}$, verify that $AB = I_2$ but $BA \neq I_3$.

14. Prove Theorem 1 for elementary matrices of: (a) type I; (b) type II.

15. While trying to invert A, $[A \ \ I]$ is carried to $[P \ \ Q]$ by row operations. Show that $P = QA$.

16. If A and B are $n \times n$ matrices and AB is a product of elementary matrices, show that the same is true of A.

♦17. If U is invertible, show that the reduced row-echelon form of a matrix $[U \ \ A]$ is $[I \ \ U^{-1}A]$.

18. Two matrices A and B are called **row-equivalent** (written $A \overset{r}{\sim} B$) if there is a sequence of elementary row operations carrying A to B.

(a) Show that $A \overset{r}{\sim} B$ if and only if $A = UB$ for some invertible matrix U.

♦(b) Show that:

(i) $A \overset{r}{\sim} A$ for all matrices A.

(ii) If $A \overset{r}{\sim} B$, then $B \overset{r}{\sim} A$.

(iii) If $A \overset{r}{\sim} B$ and $B \overset{r}{\sim} C$, then $A \overset{r}{\sim} C$.

(c) Show that, if A and B are both row-equivalent to some third matrix, then $A \overset{r}{\sim} B$.

(d) Show that $\begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & 4 & 1 \\ 1 & 0 & 8 & 6 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 & 4 & 5 \\ -2 & 1 & -11 & -8 \\ -1 & 2 & 2 & 2 \end{bmatrix}$ are row-equivalent. [*Hint:* Consider part (c) and Theorem 1 §1.2.]

19. If U and V are invertible $n \times n$ matrices, show that $U \overset{r}{\sim} V$. (See Exercise 18.)

20. (See Exercise 18). Find all matrices that are row-equivalent to:

(a) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ♦(b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

21. Let A and B be $m \times n$ and $n \times m$ matrices, respectively. If $m > n$, show that AB is not invertible. [*Hint:* Use Theorem 1 §1.3 to find $X \neq 0$ with $BX = 0$.]

22. Define an *elementary column operation* on a matrix to be one of the following: (I) Interchange

two columns. (II) Multiply a column by a nonzero scalar. (III) Add a multiple of a column to another column. Show that:

(a) If an elementary column operation is done to an $m \times n$ matrix A, the result is AF, where F is an $n \times n$ elementary matrix.

(b) Given any $m \times n$ matrix A, there exist $m \times m$ elementary matrices $E_1, \dots, E_k$ and $n \times n$ elementary matrices $F_1, \dots, F_p$ such that, in block form,

$$E_k \cdots E_1 A F_1 \cdots F_p = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

23. Suppose B is obtained from A by:
    (a) interchanging rows i and j;
    ♦(b) multiplying row i by $k \neq 0$;
    (c) adding k times row i to row j ($i \neq j$).

    In each case describe how to obtain B^{-1} from A^{-1}. [*Hint:* See part (a) of the preceding exercise.]

Section 2.5 LU-Factorization[6]

In this section the Gaussian algorithm is used to show that any matrix A can be written as a product of matrices of a particularly nice type. This is used in computer programs to solve systems of linear equations.

An $m \times n$ matrix A is called **upper triangular** if each entry of A below and to the left of the main diagonal is zero. Here, as for square matrices, the elements $a_{11}, a_{22}, \dots$ are called the **main diagonal** of A. Hence, the matrices

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -3 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 2 & 1 & 0 & 5 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are all upper triangular. Note that each row-echelon matrix is upper triangular.

By analogy a matrix is called **lower triangular** if its transpose is upper triangular—that is, each entry above and to the right of the main diagonal is zero. A matrix is called **triangular** if it is either upper or lower triangular.

One reason for the importance of triangular matrices is the ease with which systems of linear equations can be solved when the coefficient matrix is triangular.

Example 1

Solve the system

$$\begin{aligned} x_1 + 2x_2 - 3x_3 - x_4 + 5x_5 &= 3 \\ 5x_3 + x_4 + x_5 &= 8 \\ 2x_5 &= 6 \end{aligned}$$

where the coefficient matrix is upper triangular.

Solution

As for a row-echelon matrix, let $x_2 = s$ and $x_4 = t$. Then solve for x_5, x_3, and x_1 in that order as follows:

$$x_5 = \tfrac{6}{2} = 3$$

Substitution into the second equation gives

$$x_3 = 1 - \tfrac{1}{5}t$$

Finally, substitution of both x_5 and x_3 into the first equation gives

$$x_1 = -9 - 2s + \tfrac{2}{5}t$$

[6]This section is not used later, so it may be omitted with no loss of continuity.

The method used in Example 1 is called **back substitution** for obvious reasons. It works because the matrix is upper triangular, and it provides an efficient method for finding the solutions (when they exist). In particular, it can be used in Gaussian elimination because row-echelon matrices are upper triangular. Similarly, if the matrix of a system of equations is lower triangular, the system can be solved (if a solution exists) by **forward substitution**. Here each equation is used to solve for one variable by substituting values already found for earlier variables.

Suppose now that an arbitrary matrix A is given and consider the system

$$AX = B$$

of linear equations with A as the coefficient matrix. If A can be factored as $A = LU$, where L is lower triangular and U is upper triangular, the system can be solved in two stages as follows:

1. Solve $LY = B$ for Y by forward substitution.
2. Solve $UX = Y$ for X by back substitution.

Then X is a solution to $AX = B$ because $AX = LUX = LY = B$. Moreover, every solution arises in this way (take $Y = UX$). This focuses attention on obtaining such factorizations $A = LU$ of a matrix A.

The Gaussian algorithm provides a method of obtaining these factorizations. The method exploits the following facts about triangular matrices.

Lemma 1

The product of two lower triangular matrices (or two upper triangular matrices) is again lower triangular (upper triangular).

Lemma 2

Let A be an $n \times n$ lower triangular (or upper triangular) matrix. Then A is invertible if and only if no main diagonal entry is zero. In this case, A^{-1} is also lower (upper) triangular.

The proofs are straightforward and are left as Exercises 8 and 9.

Now let A be any $m \times n$ matrix. The Gaussian algorithm produces a sequence of row operations that carry A to a row-echelon matrix U. However, no multiple of a row is ever added to a row *above* it (because we are not insisting on *reduced* row-echelon form). The point is that, apart from row interchanges,[7] the only row operations needed are those that make the corresponding elementary matrix *lower triangular*. This observation gives the following theorem.

Theorem 1

Suppose that, via the Gaussian algorithm, a matrix A can be carried to a row-echelon matrix U using no row interchanges. Then

$$A = LU$$

where L is lower triangular and invertible and U is row-echelon (and upper triangular).

[7] Any row interchange can actually be accomplished by row operations of other types (Exercise 6), but one of these must involve adding a multiple of some row to a row *above* it.

Proof

The hypotheses imply that there exist lower triangular, elementary matrices $E_1, E_2, \dots, E_k$ such that $U = (E_k \cdots E_2 E_1)A$. Hence $A = LU$, where $L = E_1^{-1}E_2^{-1} \cdots E_k^{-1}$ is lower triangular and invertible by Lemmas 1 and 2.

A factorization $A = LU$ as in Theorem 1 is called an **LU-factorization** of the matrix A. Such a factorization may not exist (Exercise 4) because at least one row interchange is required in the Gaussian algorithm. A procedure for dealing with this situation will be outlined later. However, if an LU-factorization does exist, the row-echelon matrix U in Theorem 1 is obtained by Gaussian elimination and the algorithm also yields a simple procedure for writing down the matrix L. The following example illustrates the technique.

Example 2

Find an LU-factorization of the matrix $A = \begin{bmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & -1 & 3 & 3 & 2 \\ 0 & -1 & 3 & 7 & 10 \end{bmatrix}$.

Solution

We are assuming that we can carry A to a row-echelon matrix U as before, using no row interchanges. The steps in the Gaussian algorithm are shown, and at each stage the corresponding elementary matrix is computed. The reason for the circled entries will be apparent shortly.

$$\begin{bmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & -1 & 3 & 3 & 2 \\ 0 & -1 & 3 & 7 & 10 \end{bmatrix} = A$$

$$\begin{bmatrix} 0 & 1 & -3 & -1 & 2 \\ 0 & -1 & 3 & 3 & 2 \\ 0 & -1 & 3 & 7 & 10 \end{bmatrix} = E_1A \qquad E_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -3 & -1 & 2 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & -1 & 3 & 7 & 10 \end{bmatrix} = E_2E_1A \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -3 & -1 & 2 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 6 & 12 \end{bmatrix} = E_3E_2E_1A \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -3 & -1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 6 & 12 \end{bmatrix} = E_4E_3E_2E_1A \qquad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & 1 & -3 & -1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = E_5E_4E_3E_2E_1A \qquad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{bmatrix}$$

Thus (as in the proof of Theorem 1), the LU-factorization of A is $A = LU$, where

$$L = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 6 & 1 \end{bmatrix}$$

Now observe that the first two columns of L can be obtained from the columns circled during execution of the algorithm.

The procedure in Example 2 works in general. Moreover, we can construct the matrix L as we go along, one (circled) column at a time, starting from the left. There is no need to compute the elementary matrices E_k, and the method is suitable for use in a computer program because the circled columns can be stored in memory as they are created.

To describe the process in general, the following notation is useful. Given positive integers $m \geq r$, let $C_1, C_2, \dots, C_r$ be columns of decreasing lengths $m, m-1, \dots, m-r+1$. Then let

$$L_m[C_1, \dots, C_r] \qquad (*)$$

denote the $m \times m$ lower triangular matrix obtained from the identity matrix by replacing the bottom $m-j+1$ entries of column j by C_j for each $j = 1, 2, \dots, r$.

Thus, the matrix L in Example 2 has this form:

$$L = L_3[C_1, C_2] = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 6 & 1 \end{bmatrix} \quad \text{where } C_1 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \text{ and } C_2 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}.$$

Here is another example.

Example 3

If $C_1 = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 2 \end{bmatrix}$ and $C_2 = \begin{bmatrix} 5 \\ -1 \\ 7 \end{bmatrix}$, then $L_4[C_1, C_2] = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -1 & 5 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix}$.

Note that if $r < m$, the last $m - r$ columns of $L_m[C_1, \dots, C_r]$ are the corresponding columns of the identity matrix I_m.

Now the general version of the procedure in Example 2 can be stated. Given a nonzero matrix A, call the first nonzero column of A (from the left) the **leading column** of A.

LU-Algorithm

Let A be an $m \times n$ matrix that can be carried to a row-echelon matrix U using no row interchanges. An LU-factorization $A = LU$ can be obtained as follows:

Step 1. If $A = 0$, take $L = I_m$ and $U = 0$.

Step 2. If $A \neq 0$, let C_1 be the leading column of A and do row operations (with no row interchanges) to create the first leading 1 and bring A to the following block form:

$$A \to \left[\begin{array}{c|c|c} 0 & 1 & X_2 \\ \hline 0 & 0 & A_2 \end{array}\right]$$

Step 3. If $A_2 \neq 0$, let C_2 be the leading column of A_2 and apply step 2 to bring A_2 to block form:

$$A_2 \to \left[\begin{array}{c|c|c} 0 & 1 & X_3 \\ \hline 0 & 0 & A_3 \end{array}\right]$$

Step 4. Continue in this way until all the rows below the last leading 1 created consist of zeros. Take U to be the (row-echelon) matrix just created, and take [see equation ($*$) on page 71]

$$L = L_m[C_1, C_2, \dots, C_r]$$

where $C_1, C_2, C_3, \dots$ are the leading columns of the matrices $A, A_2, A_3, \dots$.

The proof is given at the end of this section.

Of course, the integer r in the LU-algorithm is the number of leading 1's in the row-echelon matrix U, so it is the rank of A.

Example 4

Find an LU-factorization for $A = \begin{bmatrix} 5 & -5 & 10 & 0 & 5 \\ -3 & 3 & 2 & 2 & 1 \\ -2 & 2 & 0 & -1 & 0 \\ 1 & -1 & 10 & 2 & 5 \end{bmatrix}$.

Solution

The reduction to row-echelon form is

$$\begin{bmatrix} 5 & -5 & 10 & 0 & 5 \\ -3 & 3 & 2 & 2 & 1 \\ -2 & 2 & 0 & -1 & 0 \\ 1 & -1 & 10 & 2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 8 & 2 & 4 \\ 0 & 0 & 4 & -1 & 2 \\ 0 & 0 & 8 & 2 & 4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

If U denotes this row-echelon matrix, then $A = LU$, where

$$L = \begin{bmatrix} 5 & 0 & 0 & 0 \\ -3 & 8 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 1 & 8 & 0 & 1 \end{bmatrix}$$

The next example deals with a case where no row of zeros is present in U (in fact, A is invertible).

Example 5

Find an LU-factorization for $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix}$.

Solution

The reduction to row-echelon form is

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

so $L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & 5 \end{bmatrix}$.

There are matrices (for example $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$) that have no LU-factorization and so require at least one row interchange when being carried to row-echelon form via the Gaussian algorithm. However, it turns out that if all the row interchanges encountered in the algorithm are carried out first, the resulting matrix requires no interchanges and so has an LU-factorization. Here is the precise result.

Theorem 2

Suppose an $m \times n$ matrix A is carried to a row-echelon matrix U via the Gaussian algorithm. Let $P_1, P_2, \dots, P_s$ be the elementary matrices corresponding (in order) to the row interchanges used and write $P = P_s \cdots P_2 P_1$. (If no interchanges are used take $P = I_m$.) Then:

1. PA is the matrix obtained from A by doing these interchanges (in order) to A.
2. PA has an LU-factorization.

The proof is given at the end of this section.

A matrix P that is the product of elementary matrices corresponding to row interchanges is called a **permutation matrix**. Such a matrix is obtained from the identity matrix by arranging the rows in a different order, so it has exactly one 1 in each row and each column, and has zeros elsewhere. We regard the identity matrix as a permutation matrix. The elementary permutation matrices are those obtained from I by a single row interchange, and every permutation matrix is a product of elementary ones.

Example 6

If $A = \begin{bmatrix} 0 & 0 & -1 & 2 \\ -1 & -1 & 1 & 2 \\ 2 & 1 & -3 & 6 \\ 0 & 1 & -1 & 4 \end{bmatrix}$, find a permutation matrix P such that PA has an LU-factorization, and then find the factorization.

Solution

Apply the Gaussian algorithm to A:

$$A \to \begin{bmatrix} -1 & -1 & 1 & 2 \\ 0 & 0 & -1 & 2 \\ 2 & 1 & -3 & 6 \\ 0 & 1 & -1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 0 & -1 & 2 \\ 0 & -1 & -1 & 10 \\ 0 & 1 & -1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & -1 & -1 & 10 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -2 & 14 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

Two row interchanges were needed, first rows 1 and 2 and then rows 2 and 3. Hence, as in Theorem 2,

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If we do these interchanges (in order) to A, the result is PA. Now apply the LU-algorithm to PA:

$$PA = \begin{bmatrix} -1 & -1 & 1 & 2 \\ 2 & 1 & -3 & 6 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & -1 & -1 & 10 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -2 & 14 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U$$

Hence, $PA = LU$, where $L = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 10 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Theorem 2 provides an important general factorization theorem for matrices. If A is any $m \times n$ matrix, it asserts that there exists a permutation matrix P and an LU-factorization $PA = LU$. Moreover, it shows that either $P = I$ or $P = P_s \cdots P_2P_1$, where $P_1, P_2, \dots, P_s$ are the elementary permutation matrices arising in the reduction of A to row-echelon form. Now observe that $P_i^{-1} = P_i$ for each i.

Thus, $P^{-1} = P_1P_2 \cdots P_s$, so the matrix A can be factored as

$$A = P^{-1}LU$$

where P^{-1} is a permutation matrix, L is lower triangular and invertible, and U is a row-echelon matrix. This is called a **PLU-factorization** of A.

The LU-factorization in Theorem 1 is not unique. For example,

$$\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

However, the fact that the row-echelon matrix here has a row of zeros is necessary. Recall that the rank of a matrix A is the number of nonzero rows in any row-echelon matrix U to which A can be carried by row operations. Thus, if A is $m \times n$, the matrix U has no row of zeros if and only if A has rank m.

Theorem 3

Let A be an $m \times n$ matrix that has an LU-factorization

$$A = LU$$

If A has rank m (that is, U has no row of zeros), then L and U are uniquely determined by A.

Proof

Suppose $A = MV$ is another LU-factorization of A, so M is lower triangular and invertible and V is row-echelon. Hence $LU = MV$, and we must show that $L = M$ and $U = V$. We write $N = M^{-1}L$. Then N is lower triangular and invertible (Lemmas 1 and 2) and $NU = V$, so it suffices to prove that $N = I$. If N is $m \times m$, we use induction on m. The case $m = 1$ is left to the reader. If $m > 1$, observe first that column 1 of V is N times column 1 of U. Thus if either column is zero, so is the other (N is invertible). Hence, we can assume (by deleting zero columns) that the (1, 1)-entry is 1 in both U and V.

Now we write $N = \begin{bmatrix} a & 0 \\ X & N_1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & Y \\ 0 & U_1 \end{bmatrix}$, and $V = \begin{bmatrix} 1 & Z \\ 0 & V_1 \end{bmatrix}$ in block form.

Then $NU = V$ becomes $\begin{bmatrix} a & aY \\ X & XY + N_1U_1 \end{bmatrix} = \begin{bmatrix} 1 & Z \\ 0 & V_1 \end{bmatrix}$. Hence $a = 1$, $Y = Z$, $X = 0$, and $N_1U_1 = V_1$. But $N_1U_1 = V_1$ implies $N_1 = I$ by induction, whence $N = I$.

If A is an $m \times m$ invertible matrix, then A has rank m by Theorem 5 §2.3. Hence, we get the following important special case of Theorem 3.

Corollary

If an invertible matrix A has an LU-factorization $A = LU$, then L and U are uniquely determined by A.

Of course, in this case U is an upper triangular matrix with 1s along the main diagonal.

Proofs of Theorems

Proof of the LU-algorithm

Proceed by induction on n. If $n = 1$, it is left to the reader. If $n > 1$, let C_1 denote the leading column of A and let K_1 denote the first column of the $m \times m$ identity matrix. There exist elementary matrices $E_1, \dots, E_k$ such that, in block form,

$$(E_k \cdots E_2E_1)A = \left[\begin{array}{c|c|c} 0 & K_1 & \begin{array}{c} X_1 \\ \hline A_1 \end{array} \end{array}\right] \quad \text{where } (E_k \cdots E_2E_1)C_1 = K_1.$$

Moreover, each E_j can be taken to be lower triangular (by assumption). Write

$$L_0 = (E_k \cdots E_2E_1)^{-1} = E_1^{-1}E_2^{-1} \cdots E_k^{-1}$$

Then L_0 is lower triangular, and $L_0K_1 = C_1$. Also, each E_j (and so each E_j^{-1}) is the result of either multiplying row 1 of I_m by a constant or adding a multiple of row 1 to another row. Hence,

$$L_0 = (E_1^{-1}E_2^{-1} \cdots E_k^{-1})I_m = \left[\begin{array}{c|c} C_1 & \begin{array}{c} 0 \\ \hline I_{m-1} \end{array} \end{array}\right]$$

in block form. Now, by induction, let $A_1 = L_1U_1$ be an LU-factorization of A_1, where $L_1 = L_{m-1}[C_2, \dots, C_r]$ and U_1 is row-echelon. Then block multiplication gives

$$L_0^{-1}A = \left[\begin{array}{c|c|c} 0 & K_1 & \begin{array}{c} X_1 \\ \hline L_1U_1 \end{array} \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & L_1 \end{array}\right]\left[\begin{array}{c|c|c} 0 & 1 & X_1 \\ \hline 0 & 0 & U_1 \end{array}\right]$$

Hence $A = LU$, where $U = \left[\begin{array}{c|c|c} 0 & 1 & X_1 \\ \hline 0 & 0 & U_1 \end{array}\right]$ is row-echelon and

$$L = \left[\begin{array}{c|c} C_1 & \begin{array}{c} 0 \\ \hline I_{m-1} \end{array} \end{array}\right]\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & L_1 \end{array}\right] = \left[\begin{array}{c|c} C_1 & \begin{array}{c} 0 \\ \hline L_1 \end{array} \end{array}\right] = L_m[C_1, C_2, \dots, C_r]$$

This completes the proof.

Proof of Theorem 2

Let A be a nonzero $m \times n$ matrix and let K_j denote column j of I_m. There is a permutation matrix P_1 (where either P_1 is elementary or $P_1 = I_m$) such that the first nonzero column C_1 of P_1A has a nonzero entry on top. Hence, as in the LU-algorithm,

$$L_m[C_1]^{-1} \cdot P_1 \cdot A = \left[\begin{array}{c|c|c} 0 & 1 & X_1 \\ \hline 0 & 0 & A_1 \end{array}\right]$$

in block form. Then let P_2 be a permutation matrix (either elementary or I_m) such that

$$P_2 \cdot L_m[C_1]^{-1} \cdot P_1 \cdot A = \left[\begin{array}{c|c|c} 0 & 1 & X_1 \\ \hline 0 & 0 & A_1' \end{array}\right]$$

and the first nonzero column C_2 of A_1' has a nonzero entry on top. Thus,

$$L_m[K_1, C_2]^{-1} \cdot P_2 \cdot L_m[C_1]^{-1} \cdot P_1 \cdot A = \left[\begin{array}{c|c|c} 0 & 1 & X_1 \\ \hline 0 & 0 & \begin{array}{c|c|c} 0 & 1 & X_2 \\ \hline 0 & 0 & A_2 \end{array} \end{array}\right]$$

in block form. Continue to obtain elementary permutation matrices $P_1, P_2, \dots, P_r$ and columns $C_1, C_2, \dots, C_r$ of lengths $m, m-1, \dots,$ such that

$$(L_r P_r L_{r-1} P_{r-1} \cdots L_2 P_2 L_1 P_1)A = U$$

where U is a row-echelon matrix and $L_j = L_m[K_1, \dots, K_{j-1}, C_j]^{-1}$ for each j, where the notation means the first $j-1$ columns are those of I_m. It is not hard to verify that each L_j has the form $L_j = L_m[K_1, \dots, K_{j-1}, C_j']$ where C_j' is a column of length $m-j+1$. We now claim that each permutation matrix P_k can be "moved past" each matrix L_j to the right of it, in the sense that

$$P_k L_j = L_j' P_k$$

where $L_j' = L_m[K_1, \dots, K_{j-1}, C_j'']$ for some column C_j'' of length $m-j+1$. Given that this is true, we obtain a factorization of the form

$$(L_r L_{r-1}' \cdots L_2' L_1')(P_r P_{r-1} \cdots P_2 P_1)A = U$$

If we write $P = P_r P_{r-1} \cdots P_2 P_1$, this shows that PA has an LU-factorization because $L_r L_{r-1}' \cdots L_2' L_1'$ is lower triangular and invertible. All that remains is to prove the following rather technical result.

Lemma 3

Let P_k result from interchanging row k of I_m with a row below it. If $j < k$, let C_j be a column of length $m-j+1$. Then there is another column C_j' of length $m-j+1$ such that

$$P_k \cdot L_m[K_1 \cdots K_{j-1} C_j] = L_m[K_1 \cdots K_{j-1} C_j'] \cdot P_k$$

The proof is left as Exercise 12.

Exercises 2.5

1. Find an LU-factorization of the following matrices.

(a) $\begin{bmatrix} 2 & 6 & -2 & 0 & 2 \\ 3 & 9 & -3 & 3 & 1 \\ -1 & -3 & 1 & -3 & 1 \end{bmatrix}$ ♦(b) $\begin{bmatrix} 2 & 4 & 2 \\ 1 & -1 & 3 \\ -1 & 7 & -7 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 6 & -2 & 0 & 2 \\ 1 & 5 & -1 & 2 & 5 \\ 3 & 7 & -3 & -2 & 5 \\ -1 & -1 & 1 & 2 & 3 \end{bmatrix}$

♦(d) $\begin{bmatrix} -1 & -3 & 1 & 0 & -1 \\ 1 & 4 & 1 & 1 & 1 \\ 1 & 2 & -3 & -1 & 1 \\ 0 & -2 & -4 & -2 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & 2 & 4 & 6 & 0 & 2 \\ 1 & -1 & 2 & 1 & 3 & 1 \\ -2 & 2 & -4 & -1 & 1 & 6 \\ 0 & 2 & 0 & 3 & 4 & 8 \\ -2 & 4 & -4 & 1 & -2 & 6 \end{bmatrix}$ ♦(f) $\begin{bmatrix} 2 & 2 & -2 & 4 & 2 \\ 1 & -1 & 0 & 2 & 1 \\ 3 & 1 & -2 & 6 & 3 \\ 1 & 3 & -2 & 2 & 1 \end{bmatrix}$

2. Find a permutation matrix P and an LU-factorization of PA if A is:

(a) $\begin{bmatrix} 0 & 0 & 2 \\ 0 & -1 & 4 \\ 3 & 5 & 1 \end{bmatrix}$ ♦(b) $\begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 4 \\ -1 & 2 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -1 & 2 & 1 & 3 \\ -1 & 1 & 3 & 1 & 4 \\ 1 & -1 & -3 & 6 & 2 \\ 2 & -2 & -4 & 1 & 0 \end{bmatrix}$ ♦(d) $\begin{bmatrix} -1 & -2 & 3 & 0 \\ 2 & 4 & -6 & 5 \\ 1 & 1 & -1 & 3 \\ 2 & 5 & -10 & 1 \end{bmatrix}$

3. In each case use the given LU-decomposition of A to solve the system $AX = B$ by finding Y such that $LY = B$, and then X such that $UX = Y$:

(a) $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

♦(b) $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$

♦(d) $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 3 & 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 4 \\ -6 \\ 4 \\ 5 \end{bmatrix}$

4. Show that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = LU$ is impossible where L is lower triangular and U is upper triangular.

5. Let E and F be the elementary matrices obtained from the identity matrix by adding multiples of row k to rows p and q. If $k \neq p$ and $k \neq q$, show that $EF = FE$.

♦6. Show that we can accomplish any row interchange by using only row operations of other types.

7. (a) Let L and L_1 be invertible lower triangular matrices, and let U and U_1 be invertible upper triangular matrices. Show that $LU = L_1U_1$ if and only if there exists an invertible diagonal matrix D such that $L_1 = LD$ and $U_1 = D^{-1}U$. [*Hint:* Scrutinize $L^{-1}L_1 = UU_1^{-1}$.]

♦(b) Use part (a) to prove Theorem 3 in the case that A is invertible.

♦8. Prove Lemma 1. [*Hint:* Use block multiplication and induction.]

9. Prove Lemma 2. [*Hint:* Use block multiplication and induction.]

10. A triangular matrix is called **unit triangular** if it is square and every main diagonal element is a 1.

(a) If A can be carried by the Gaussian algorithm to row-echelon form using no row interchanges, show that $A = LU$ where L is unit lower triangular and U is upper triangular.

(b) Show that the factorization in (a) is unique.

11. Let $C_1, C_2, \dots, C_r$ be columns of lengths $m, m-1, \dots, m-r+1$. If K_j denotes column j of I_m, show that $L_m[C_1, C_2, \dots, C_r] = L_m[C_1]L_m[K_1, C_2]L_m[K_1, K_2, C_3] \cdots L_m[K_1, K_2, \dots, K_{r-1}, C_r]$. The notation is as in the proof of Theorem 2. [*Hint:* Use induction on m and block multiplication.]

12. Prove Lemma 3. [*Hint:* $P_k^{-1} = P_k$. Write $P_k = \begin{bmatrix} I_k & 0 \\ 0 & P_0 \end{bmatrix}$ in block form where P_0 is an $(m-k) \times (m-k)$ permutation matrix.]

Section 2.6 An Application to Input-Output Economic Models[8]

In 1973 Wassily Leontief was awarded the Nobel prize in economics for his work on mathematical models.[9] Roughly speaking, an economic system in this model consists of several industries, each of which produces a product and each of which uses some of the production of the other industries. The following example is typical.

Example 1

A primitive society has three basic needs: food, shelter, and clothing. There are thus three industries in the society—the farming, housing, and garment industries—that produce these commodities. Each of these industries consumes a certain proportion of the total output of each commodity according to the following table.

		OUTPUT		
		Farming	**Housing**	**Garment**
	Farming	0.4	0.2	0.3
CONSUMPTION	**Housing**	0.2	0.6	0.4
	Garment	0.4	0.2	0.3

Find the annual prices that each industry must charge for its income to equal its expenditures.

Solution

Let p_1, p_2, and p_3 be the prices charged per year by the farming, housing, and garment industries, respectively, for their total output. To see how these prices are determined, consider the farming industry. It receives p_1 for its production in any year. But it *consumes* products from all these industries in the following amounts (from row 1 of the table): 40% of the food, 20% of the housing, and 30% of the clothing. Hence, the expenditures of the farming industry are $0.4p_1 + 0.2p_2 + 0.3p_3$, so

$$0.4p_1 + 0.2p_2 + 0.3p_3 = p_1$$

A similar analysis of the other two industries leads to the following system of equations.

$$\begin{aligned} 0.4p_1 + 0.2p_2 + 0.3p_3 &= p_1 \\ 0.2p_1 + 0.6p_2 + 0.4p_3 &= p_2 \\ 0.4p_1 + 0.2p_2 + 0.3p_3 &= p_3 \end{aligned}$$

This has the matrix form $EP = P$, where

$$E = \begin{bmatrix} 0.4 & 0.2 & 0.3 \\ 0.2 & 0.6 & 0.4 \\ 0.4 & 0.2 & 0.3 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

The equations can be written as the homogeneous system

$$(I - E)P = 0$$

where I is the 3×3 identity matrix, and the solutions are

$$P = \begin{bmatrix} 2t \\ 3t \\ 2t \end{bmatrix}$$

[8] The applications in this section and the next are independent and may be taken in any order.

[9] See W. W. Leontief, "The world economy of the year 2000," *Scientific American*, Sept. 1980.

where t is a parameter. Thus, the pricing must be such that the total output of the farming industry has the same value as the total output of the garment industry, whereas the total value of the housing industry must be $\frac{3}{2}$ as much.

In general, suppose an economy has n industries, each of which uses some (possibly none) of the production of every industry. We assume first that the economy is **closed** (that is, no product is exported or imported) and that all product is used. Given two industries i and j, let e_{ij} denote the proportion of the total annual output of industry j that is consumed by industry i. Then $E = [e_{ij}]$ is called the **input-output** matrix for the economy. Clearly,

$$0 \le e_{ij} \le 1 \quad \text{for all } i \text{ and } j \tag{1}$$

Moreover, all the output from industry j is used by *some* industry (the model is closed), so

$$e_{1j} + e_{2j} + \cdots + e_{nj} = 1 \quad \text{for each } j \tag{2}$$

Condition 2 asserts that each column of E sums to 1. Matrices satisfying conditions 1 and 2 are called **stochastic matrices**.

As in Example 1, let p_i denote the price of the total annual production of industry i. Then p_i is the annual revenue of industry i. On the other hand, industry i spends $e_{i1}p_1 + e_{i2}p_2 + \cdots + e_{in}p_n$ annually for the product it uses ($e_{ij}p_j$ is the cost for product from industry j). The closed economic system is said to be in **equilibrium** if the annual expenditure equals the annual revenue for each indusry—that is, if

$$e_{i1}p_1 + e_{i2}p_2 + \cdots + e_{in}p_n = p_i \quad \text{for each } i = 1, 2, \dots, n$$

If we write $P = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$, these equations can be written as the matrix equation

$$EP = P$$

This is called the **equilibrium condition**, and the solutions P are called **equilibrium price structures**. The equilibrium condition can be written as

$$(I - E)P = 0$$

which is a system of homogeneous equations for P. Moreover, there is always a non-trivial solution P. Indeed, the column sums of $I - E$ are all 0 (because E is stochastic), so the row-echelon form of $I - E$ has a row of zeros. In fact, more is true:

Theorem 1

Let E be any $n \times n$ stochastic matrix. Then there is a nonzero $n \times 1$ matrix P with nonnegative entries such that $EP = P$. If all the entries of E are positive, the matrix P can be chosen with all entries positive.

Theorem 1 guarantees the existence of an equilibrium price structure for any closed input-output system of the type discussed here. The proof is beyond the scope of this book.[10]

[10] The interested reader is referred to P. Lancaster's *Theory of Matrices* (New York: Academic Press, 1969) or to E. Seneta's *Non-negative Matrices* (New York: Wiley, 1973).

Example 2

Find the equilibrium price structures for four industries if the input-output matrix is

$$E = \begin{bmatrix} .6 & .2 & .1 & .1 \\ .3 & .4 & .2 & 0 \\ .1 & .3 & .5 & .2 \\ 0 & .1 & .2 & .7 \end{bmatrix}$$

Find the prices if the total value of business is \$1000.

Solution

If $P = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$ is the equilibrium price structure, then the equilibrium condition is $EP = P$. When we write this as $(I - E)P = 0$, the methods of Chapter 1 yield the following family of solutions:

$$P = \begin{bmatrix} 44t \\ 39t \\ 51t \\ 47t \end{bmatrix}$$

where t is a parameter. If we insist that $p_1 + p_2 + p_3 + p_4 = 1000$, then $t = 5.525$ (to four figures). Hence

$$P = \begin{bmatrix} 243.09 \\ 215.47 \\ 281.77 \\ 259.67 \end{bmatrix}$$

to five figures.

The Open Model

We now assume that there is a demand for products in the **open sector** of the economy, which is the part of the economy other than the producing industries (for example, consumers). Let d_i denote the total value of the demand for product i in the open sector. If p_i and e_{ij} are as before, the value of the annual demand for product i by the producing industries themselves is $e_{i1}p_1 + e_{i2}p_2 + \cdots + e_{in}p_n$, so the total annual revenue p_i of industry i breaks down as follows:

$$p_i = (e_{i1}p_1 + e_{i2}p_2 + \cdots + e_{in}p_n) + d_i \quad \text{for each } i = 1, 2, \ldots, n$$

The column $D = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$ is called the **demand matrix**, and this gives a matrix equation

$$P = EP + D$$

or

$$(I - E)P = D \qquad (*)$$

This is a system of linear equations for P, and we ask for a solution P with every entry nonnegative. Note that every entry of E is between 0 and 1, but the column sums of E need not equal 1 as in the closed model.

Before proceeding, it is convenient to introduce a useful notation. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of the same size, we write $A > B$ if $a_{ij} > b_{ij}$ for all i and j, and we write $A \geq B$ if $a_{ij} \geq b_{ij}$ for all i and j. Thus $P \geq 0$ means that every entry of P is nonnegative. Note that $A \geq 0$ and $B \geq 0$ implies that $AB \geq 0$.

Now, given a demand matrix $D \geq 0$, we look for a production matrix $P \geq 0$ satisfying equation (∗). This certainly exists if $I - E$ is invertible and $(I - E)^{-1} \geq 0$. On the other hand, the fact that $D \geq 0$ means any solution P to equation (∗) satisfies $P \geq EP$. Hence, the following theorem is not too surprising.

Theorem 2

Let $E \geq 0$ be a square matrix. Then $I - E$ is invertible and $(I - E)^{-1} \geq 0$ if and only if there exists a column $P > 0$ such that $P > EP$.

Heuristic Proof

If $(I - E)^{-1} \geq 0$, the existence of $P > 0$ with $P > EP$ is left as Exercise 11. Conversely, suppose such a column P exists. Observe that

$$(I - E)(I + E + E^2 + \cdots + E^{k-1}) = I - E^k$$

holds for all $k \geq 2$. If we can show that every entry of E^k approaches 0 as k becomes large then, intuitively, the infinite matrix sum

$$U = I + E + E^2 + \cdots$$

exists and $(I - E)U = I$. Since $U \geq 0$, this does it. To show that E^k approaches 0, it suffices to show that $EP < \mu P$ for some number μ with $0 < \mu < 1$ (then $E^k P < \mu^k P$ for all $k \geq 1$ by induction). The existence of μ is left as Exercise 12.

The condition $P > EP$ in Theorem 2 has a simple economic interpretation. If P is a production matrix, entry i of EP is the total value of all product used by industry i in a year. Hence, the condition $P > EP$ means that, for each i, the value of product produced by industry i exceeds the value of the product it uses. In other words, each industry runs at a profit.

Example 3

If $E = \begin{bmatrix} 0.6 & 0.2 & 0.3 \\ 0.1 & 0.4 & 0.2 \\ 0.2 & 0.5 & 0.1 \end{bmatrix}$, show that $I - E$ is invertible and $(I - E)^{-1} \geq 0$.

Solution

Use $P = [3 \;\; 2 \;\; 2]^T$ in Theorem 2.

If $P_0 = [1 \;\; 1 \;\; \cdots \;\; 1]^T$, the entries of EP_0 are the row sums of E. Hence $P_0 > EP_0$ holds if the row sums of E are all less than 1. This proves the first of the following useful facts (the second is Exercise 10).

Corollary

Let $E \geq 0$ be a square matrix. In each of the following cases, $I - E$ is invertible and $(I - E)^{-1} \geq 0$:

1. All row sums of E are less than 1.
2. All column sums of E are less than 1.

Exercises 2.6

1. Find the possible equilibrium price structures when the input-output matrices are:

   (a) $\begin{bmatrix} .1 & .2 & .3 \\ .6 & .2 & .3 \\ .3 & .6 & .4 \end{bmatrix}$ ♦(b) $\begin{bmatrix} .5 & 0 & .5 \\ .1 & .9 & .2 \\ .4 & .1 & .3 \end{bmatrix}$

   (c) $\begin{bmatrix} .3 & .1 & .1 & .2 \\ .2 & .3 & .1 & 0 \\ .3 & .3 & .2 & .3 \\ .2 & .3 & .6 & .5 \end{bmatrix}$ ♦(d) $\begin{bmatrix} .5 & 0 & .1 & .1 \\ .2 & .7 & 0 & .1 \\ .1 & .2 & .8 & .2 \\ .2 & .1 & .1 & .6 \end{bmatrix}$

♦2. Three industries A, B, and C are such that all the output of A is used by B, all the output of B is used by C, and all the output of C is used by A. Find the possible equilibrium price structures.

3. Find the possible equilibrium price structures for three industries where the input-output matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Discuss why there are two parameters here.

♦4. Prove Theorem 1 for a 2×2 stochastic matrix E by first writing it in the form $E = \begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix}$, where $0 \le a \le 1$ and $0 \le b \le 1$.

5. If E is an $n \times n$ stochastic matrix and C is an $n \times 1$ matrix, show that the sum of the entries of C equals the sum of the entries of the $n \times 1$ matrix EC.

6. Let $W = [1 \ 1 \ 1 \ \cdots \ 1]$. Let E and F denote $n \times n$ matrices with nonnegative entries.

   (a) Show that E is a stochastic matrix if and only if $WE = W$.

   (b) Use part (a) to deduce that, if E and F are both stochastic matrices, then EF is also stochastic.

7. Find a 2×2 matrix E with entries between 0 and 1 such that:

   (a) $I - E$ has no inverse.

   ♦(b) $I - E$ has an inverse but not all entries of $(I - E)^{-1}$ are nonnegative.

♦8. If E is a 2×2 matrix with entries between 0 and 1, show that $I - E$ is invertible and $(I - E)^{-1} \ge 0$ if and only if tr $E < 1 + \det E$. Here, if $E = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then tr $E = a + d$ and $\det E = ad - bc$.

9. In each case show that $I - E$ is invertible and $(I - E)^{-1} \ge 0$.

   (a) $\begin{bmatrix} .6 & .5 & .1 \\ .1 & .3 & .3 \\ .2 & .1 & .4 \end{bmatrix}$ ♦(b) $\begin{bmatrix} .7 & .1 & .3 \\ .2 & .5 & .2 \\ .1 & .1 & .4 \end{bmatrix}$

   (c) $\begin{bmatrix} .6 & .2 & .1 \\ .3 & .4 & .2 \\ .2 & .5 & .1 \end{bmatrix}$ (d) $\begin{bmatrix} .8 & .1 & .1 \\ .3 & .1 & .2 \\ .3 & .3 & .2 \end{bmatrix}$

10. Prove that (1) implies (2) in the Corollary to Theorem 2.

11. If $(I - E)^{-1} \ge 0$, find $P > 0$ such that $P > EP$.

12. If $EP < P$ where $E \ge 0$ and $P > 0$, find a number μ such that $EP < \mu P$ and $0 < \mu < 1$. [*Hint:* If $EP = [q_1, \dots, q_n]^T$ and $P = [p_1, \dots, p_n]^T$, take any number μ such that $\max\{\frac{q_1}{p_1}, \dots, \frac{q_n}{p_n}\} < \mu < 1$.]

Section 2.7 An Application to Markov Chains

Many natural phenomena progress through various stages and can be in a variety of states at each stage. For example, the weather in a given city progresses day by day and, on any given day, may be sunny or rainy. Here the states are "sun" and "rain," and the weather progresses from one state to another in daily stages. Another example might be a football team: The stages of its evolution are the games it plays, and the possible states are "win," "draw," and "loss."

The general setup is as follows: A "system" evolves through a series of "stages," and at any stage it can be in any one of a finite number of "states." At any given stage, the state to which it will go at the next stage depends on the past and present history of the system—that is, on the sequence of states it has occupied to date. A **Markov chain** is such an evolving system wherein the state to which it will go next depends *only* on its *present* state and does not depend on the earlier history of the system.[11]

[11] The name honors Andrei Andreyevich Markov (1856–1922) who was a professor at the university in St. Petersburg, Russia.

Even in the case of a Markov chain, the state the system will occupy at any stage is determined only in terms of probabilities. In other words, chance plays a role. For example, if a football team wins a particular game, we do not know whether it will win, draw, or lose the next game. On the other hand, we may know that the team tends to persist in winning streaks; for example, if it wins one game it may win the next game $\frac{1}{2}$ of the time, lose $\frac{4}{10}$ of the time, and draw $\frac{1}{10}$ of the time. These fractions are called the **probabilities** of these various possibilities. Similarly, if the team loses, it may lose the next game with probability $\frac{1}{2}$ (that is, half the time), win with probability $\frac{1}{4}$, and draw with probability $\frac{1}{4}$. The probabilities of the various outcomes after a drawn game will also be known.

We shall treat probabilities informally here: *The probability that a given event will occur is the long-run proportion of the time that the event does indeed occur*. Hence, all probabilities are numbers between 0 and 1. A probability of 0 means the event is impossible and never occurs; events with probability 1 are certain to occur.

If a Markov chain is in a particular state, the probabilities that it goes to the various states at the next stage of its evolution are called the **transition probabilities** for the chain, and they are assumed to be known quantities. To motivate the general conditions that follow, consider the following simple example. Here the system is a man, the stages are his successive lunches, and the states are the two restaurants he chooses.

Example 1

A man always eats lunch at one of two restaurants, A and B. He never eats at A twice in a row. However, if he eats at B, he is three times as likely to eat at B next time as at A. Initially, he is equally likely to eat at either restaurant.

(a) What is the probability that he eats at A on the third day after the initial one?

(b) What proportion of his lunches does he eat at A?

Solution

The table of transition probabilities follows. The A column indicates that if he eats at A on one day, he never eats there again on the next day and so is certain to go to B.

		PRESENT LUNCH	
		A	***B***
NEXT LUNCH	***A***	0	0.25
	B	1	0.75

The B column shows that, if he eats at B on one day, he will eat there on the next day $\frac{3}{4}$ of the time and switches to A only $\frac{1}{4}$ of the time.

The restaurant he visits on a given day is not determined. The most that we can expect is to know the probability that he will visit A or B on that day. Let $S_m = \begin{bmatrix} s_1^{(m)} \\ s_2^{(m)} \end{bmatrix}$ denote the *state vector* for day m. Here $s_1^{(m)}$ denotes the probability that he eats at A on day m, and $s_2^{(m)}$ is the probability that he eats at B on day m. It is convenient to let S_0 correspond to the initial day. Because he is equally likely to eat at A or B on that initial day, $s_1^{(0)} = 0.5$ and $s_2^{(0)} = 0.5$, so $S_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$.

Now let

$$P = \begin{bmatrix} 0 & 0.25 \\ 1 & 0.75 \end{bmatrix}$$

denote the *transition matrix*. We claim that the relationship

$$S_{m+1} = PS_m$$

holds for all integers $m \geq 0$. This will be derived later; for now, we use it as follows to successively compute $S_1, S_2, S_3, \ldots$.

$$S_1 = PS_0 = \begin{bmatrix} 0 & .25 \\ 1 & .75 \end{bmatrix}\begin{bmatrix} .5 \\ .5 \end{bmatrix} = \begin{bmatrix} .125 \\ .875 \end{bmatrix}$$

$$S_2 = PS_1 = \begin{bmatrix} 0 & .25 \\ 1 & .75 \end{bmatrix}\begin{bmatrix} .125 \\ .875 \end{bmatrix} = \begin{bmatrix} .21875 \\ .78125 \end{bmatrix}$$

$$S_3 = PS_2 = \begin{bmatrix} 0 & .25 \\ 1 & .75 \end{bmatrix}\begin{bmatrix} .21875 \\ .78125 \end{bmatrix} = \begin{bmatrix} .1953125 \\ .8046875 \end{bmatrix}$$

Hence, the probability that his third lunch (after the initial one) is at A is approximately 0.195, whereas the probability that it is at B is 0.805.

If we carry these calculations on, the next state vectors are (to five figures)

$$S_4 = \begin{bmatrix} .20117 \\ .79883 \end{bmatrix} \qquad S_5 = \begin{bmatrix} .19971 \\ .80029 \end{bmatrix}$$

$$S_6 = \begin{bmatrix} .20007 \\ .79993 \end{bmatrix} \qquad S_7 = \begin{bmatrix} .19998 \\ .80002 \end{bmatrix}$$

Moreover, the higher values of S_m get closer and closer to $\begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$. Hence, in the long run, he eats 20% of his lunches at A and 80% at B.

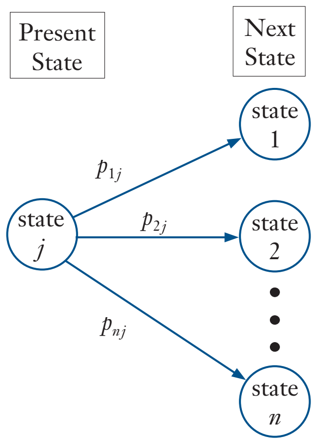


Figure 2.1

Example 1 incorporates most of the essential features of all Markov chains. The general model is as follows: The system evolves through various stages and at each stage can be in exactly one of n distinct states. It progresses through a sequence of states as time goes on. If a Markov chain is in state j at a particular stage of its development, the probability p_{ij} that it goes to state i at the next stage is called the **transition probability**. The $n \times n$ matrix $P = [p_{ij}]$ is called the **transition matrix** for the Markov chain. The situation is depicted graphically in Figure 2.1.

We make one important assumption about the transition matrix $P = [p_{ij}]$: It does *not* depend on which stage the process is in. This assumption means that the transition probabilities are *independent of time*—that is, they do not change as time goes on. It is this assumption that distinguishes Markov chains in the literature of this subject.

Example 2

Suppose the transition matrix of a three-state Markov chain is

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \begin{array}{c} \text{Present state} \\ \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} .3 & .1 & .6 \\ .5 & .9 & .2 \\ .2 & 0 & .2 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ 1 \\ 2 \\ 3 \end{array} \quad \text{Next state}$$

If, for example, the system is in state 2, column 2 lists the probabilities of where it goes next. Thus, the probability is $p_{12} = 0.1$ that it goes from state 2 to state 1, and the probability is $p_{22} = 0.9$ that it goes from state 2 to state 2. The fact that $p_{32} = 0$ means that it is impossible for it to go from state 2 to state 3 at the next stage.

Consider the jth column of the transition matrix P.

$$\begin{bmatrix} p_{1j} \\ p_{2j} \\ \vdots \\ p_{nj} \end{bmatrix}$$

If the system is in state j at some stage of its evolution, the transition probabilities $p_{1j}, p_{2j}, \ldots, p_{nj}$ represent the fraction of the time that the system will move to state 1, state 2, … , state n, respectively, at the next stage. We assume that it has to go to *some* state at each transition, so the sum of these probabilities equals 1:

$$p_{1j} + p_{2j} + \cdots + p_{nj} = 1 \quad \text{for each } j$$

Thus, the columns of P all sum to 1 and the entries of P lie between 0 and 1. A matrix with these properties is called a **stochastic matrix**.

As in Example 1, we introduce the following notation: Let $s_i^{(m)}$ denote the probability that the system is in state i after m transitions. Then $n \times 1$ matrices

$$S_m = \begin{bmatrix} s_1^{(m)} \\ s_2^{(m)} \\ \vdots \\ s_n^{(m)} \end{bmatrix} \qquad m = 0, 1, 2, \ldots$$

are called the **state vectors** for the Markov chain. Note that the sum of the entries of S_m must equal 1 because the system must be in *some* state after m transitions. The matrix S_0 is called the **initial state vector** for the Markov chain and is given as part of the data of the particular chain. For example, if the chain has only two states, then an initial vector $S_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ means that it started in state 1. If it started in state 2, the initial vector would be $S_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. If $S_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$, it is equally likely that the system started in state 1 or in state 2.

Theorem 1

Let P be the transition matrix for an n-state Markov chain. If S_m is the state vector at stage m, then

$$S_{m+1} = PS_m$$

for each $m = 0, 1, 2, \ldots$.

Heuristic Proof

Suppose that the Markov chain has been run N times, each time starting with the same initial state vector. Recall that p_{ij} is the proportion of the time the system goes from state j at some stage to state i at the next stage, whereas $s_i^{(m)}$ is the proportion of the time it is in state i at stage m. Hence

$$s_i^{(m+1)}N$$

is (approximately) the number of times the system is in state i at stage $m+1$. We are going to calculate this number another way. The system got to state i at stage $m+1$ through *some* other state (say state j) at stage m. The number of times it was *in* state j at that stage is (approximately) $s_j^{(m)}N$, so the number of times it got to state i via state j is $p_{ij}(s_j^{(m)}N)$. Summing over j gives the number of times the system is in state i (at stage $m+1$). This is the number we calculated before, so

$$s_i^{(m+1)}N = p_{i1}s_1^{(m)}N + p_{i2}s_2^{(m)}N + \cdots + p_{in}s_n^{(m)}N$$

Dividing by N gives $s_i^{(m+1)} = p_{i1}s_1^{(m)} + p_{i2}s_2^{(m)} + \cdots + p_{in}s_n^{(m)}$ for each i, and this can be expressed as the matrix equation $S_{m+1} = PS_m$.

If the initial probability vector S_0 and the transition matrix P are given, Theorem 1 gives $S_1, S_2, S_3, \ldots$, one after the other, as follows:

$$\begin{aligned} S_1 &= PS_0 \\ S_2 &= PS_1 \\ S_3 &= PS_2 \\ &\vdots \end{aligned}$$

Hence, the state vector S_m is completely determined for each $m = 0, 1, 2, \ldots$ by P and S_0.

Example 3

A wolf pack always hunts in one of three regions R_1, R_2, and R_3. Its hunting habits are as follows:

1. If it hunts in some region one day, it is as likely as not to hunt there again the next day.
2. If it hunts in R_1, it never hunts in R_2 the next day.
3. If it hunts in R_2 or R_3, it is equally likely to hunt in each of the other regions the next day.

If the pack hunts in R_1 on Monday, find the probability that it hunts there on Thursday.

Solution

The stages of this process are the successive days; the states are the three regions. The transition matrix P is determined as follows (see the table): The first habit asserts that $p_{11} = p_{22} = p_{33} = \frac{1}{2}$. Now column 1 displays what happens when the pack starts in R_1: It never goes to state 2, so $p_{21} = 0$ and, because the column must sum to 1, $p_{31} = \frac{1}{2}$. Column 2 describes what happens if it starts in R_2: $p_{22} = \frac{1}{2}$ and p_{12} and p_{32} are equal (by habit 3), so $p_{12} = p_{32} = \frac{1}{4}$ because the column sum must equal 1. Column 3 is filled in a similar way.

	R_1	R_2	R_3
R_1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
R_2	0	$\frac{1}{2}$	$\frac{1}{4}$
R_3	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$

Now let Monday be the initial stage. Then $S_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ because the pack hunts in R_1 on that day. Then S_1, S_2, and S_3 describe Tuesday, Wednesday, and Thursday, respectively, and we compute them using Theorem 1.

$$S_1 = PS_0 = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} \qquad S_2 = PS_1 = \begin{bmatrix} \frac{3}{8} \\ \frac{1}{8} \\ \frac{4}{8} \end{bmatrix} \qquad S_3 = PS_2 = \begin{bmatrix} \frac{11}{32} \\ \frac{6}{32} \\ \frac{15}{32} \end{bmatrix}$$

Hence, the probability that the pack hunts in Region R_1 on Thursday is $\frac{11}{32}$.

Another phenomenon that was observed in Example 1 can be expressed in general terms. The state vectors $S_0, S_1, S_2, \dots$ were calculated in that example and were found to "approach" $S = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$. This means that the first component of S_m becomes and remains very close to 0.2 as m becomes large, whereas the second component approaches 0.8 as m increases. When this is the case, we say that S_m **converges** to S. For large m, then, there is very little error in taking $S_m = S$, so the long-term probability that the system is in state 1 is 0.2, whereas the probability that it is in state 2 is 0.8. In Example 1, enough state vectors were computed for the limiting vector S to be apparent. However, there is a better way to do this that works in most cases.

Suppose P is the transition matrix of a Markov chain, and assume that the state vectors S_m converge to a limiting vector S. Then S_m is very close to S for sufficiently large m, so S_{m+1} is also very close to S. Thus, the equation $S_{m+1} = PS_m$ from Theorem 1 is closely approximated by

$$S = PS$$

so it is not surprising that S should be a solution to this matrix equation. Moreover, it is easily solved because it can be written as a system of linear equations

$$(I - P)S = 0$$

with the entries of S as variables.

In Example 1, where $P = \begin{bmatrix} 0 & 0.25 \\ 1 & 0.75 \end{bmatrix}$, the general solution to $(I - P)S = 0$ is $S = \begin{bmatrix} t \\ 4t \end{bmatrix}$, where t is a parameter. But if we insist that the entries of S sum to 1 (as must be true of all state vectors), we find $t = 0.2$ and so $S = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$ as before.

All this is predicated on the existence of a limiting vector for the sequence of state vectors of the Markov chain, and such a vector may not always exist. However, it does exist in one commonly occurring situation. A stochastic matrix P is called **regular** if some power P^m of P has every entry positive. The matrix $P = \begin{bmatrix} 0 & 0.25 \\ 1 & 0.75 \end{bmatrix}$ of Example 1 is regular (in this case, each entry of P^2 is positive), and the general theorem is as follows:

Theorem 2

Let P be the transition matrix of a Markov chain and assume that P is regular. Then there is a unique column matrix S satisfying the following conditions:

1. $PS = S$.
2. The entries of S are positive and sum to 1.

Moreover, condition 1 can be written as

$$(I - P)S = 0$$

and so gives a homogeneous system of linear equations for S. Finally, the sequence of state vectors $S_0, S_1, S_2, \dots$ converges to S in the sense that if m is large enough, each entry of S_m is closely approximated by the corresponding entry of S.

This theorem will not be proved here.[12]

If P is the regular transition matrix of a Markov chain, the column S satisfying conditions 1 and 2 of Theorem 2 is called the **steady-state vector** for the Markov chain. The entries of S are the long-term probabilities that the chain will be in each of the various states.

Example 4

A man eats one of three soups—beef, chicken, and vegetable—each day. He never eats the same soup two days in a row. If he eats beef soup on a certain day, he is equally likely to eat each of the others the next day; if he does not eat beef soup, he is twice as likely to eat it the next day as the alternative.

(a) If he has beef soup one day, what is the probability that he has it again two days later?

(b) What are the long-run probabilities that he eats each of the three soups?

Solution

	B	C	V
B	0	$\frac{2}{3}$	$\frac{2}{3}$
C	$\frac{1}{2}$	0	$\frac{1}{3}$
V	$\frac{1}{2}$	$\frac{1}{3}$	0

The states here are B, C, and V, the three soups. The transition matrix P is given in the table. (Recall that, for each state, the corresponding column lists the probabilities for the next state.) If he has beef soup initially, then the initial state vector is

$$S_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Then two days later the state vector is S_2. If P is the transition matrix, then

$$S_1 = PS_0 = \tfrac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad S_2 = PS_1 = \tfrac{1}{6}\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

so he eats beef soup two days later with probability $\frac{2}{3}$. This answers (a) and also shows that he eats chicken and vegetable soup each with probability $\frac{1}{6}$.

[12] The interested reader can find an elementary proof in J. Kemeny, H. Mirkil, J. Snell, and G. Thompson, *Finite Mathematical Structures* (Englewood Cliffs, N.J.: Prentice-Hall, 1958).

To find the long-run probabilities, we must find the steady-state vector S. Theorem 2 applies because P is regular (P^2 has positive entries), so S satisfies $PS = S$. That is, $(I - P)S = 0$ where

$$I - P = \frac{1}{6}\begin{bmatrix} 6 & -4 & -4 \\ -3 & 6 & -2 \\ -3 & -2 & 6 \end{bmatrix}$$

The solution is $S = \begin{bmatrix} 4t \\ 3t \\ 3t \end{bmatrix}$, where t is a parameter, and we use $S = \begin{bmatrix} 0.4 \\ 0.3 \\ 0.3 \end{bmatrix}$ because the entries of S must sum to 1. Hence, in the long run, he eats beef soup 40% of the time and eats chicken soup and vegetable soup each 30% of the time.

Exercises 2.7

1. Which of the following stochastic matrices is regular?

   (a) $\begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$ ♦(b) $\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{3} \\ \frac{1}{4} & 1 & \frac{1}{3} \\ \frac{1}{4} & 0 & \frac{1}{3} \end{bmatrix}$

2. In each case find the steady-state vector and, assuming that it starts in state 1, find the probability that it is in state 2 after 3 transitions.

   (a) $\begin{bmatrix} .5 & .3 \\ .5 & .7 \end{bmatrix}$ ♦(b) $\begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} \\ 1 & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

   ♦(d) $\begin{bmatrix} .4 & .1 & .5 \\ .2 & .6 & .2 \\ .4 & .3 & .3 \end{bmatrix}$ (e) $\begin{bmatrix} .8 & 0 & .2 \\ .1 & .6 & .1 \\ .1 & .4 & .7 \end{bmatrix}$ ♦(f) $\begin{bmatrix} .1 & .3 & .3 \\ .3 & .1 & .6 \\ .6 & .6 & .1 \end{bmatrix}$

3. A fox hunts in three territories A, B, and C. He never hunts in the same territory on two successive days. If he hunts in A, then he hunts in C the next day. If he hunts in B or C, he is twice as likely to hunt in A the next day as in the other territory.

   (a) What proportion of his time does he spend in A, in B, and in C?

   (b) If he hunts in A on Monday (C on Monday), what is the probability that he will hunt in B on Thursday?

4. Assume that there are three classes—upper, middle, and lower—and that social mobility behaves as follows:

   1. Of the children of upper-class parents, 70% remain upper-class, whereas 10% become middle-class and 20% become lower-class.

   2. Of the children of middle-class parents, 80% remain middle-class, whereas the others are evenly split between the upper class and the lower class.

   3. For the children of lower-class parents, 60% remain lower-class, whereas 30% become middle-class and 10% upper-class.

   (a) Find the probability that the grandchild of lower-class parents becomes upper-class.

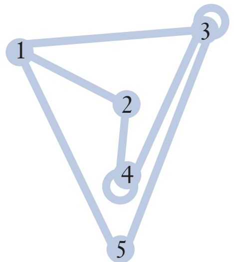
   ♦(b) Find the long-term breakdown of society into classes.

5. The Prime Minister says she will call an election. This gossip is passed from person to person with a probability $p \neq 0$ that the information is passed incorrectly at any stage. Assume that when a person hears the gossip he or she passes it to one person who does not know. Find the long-term probability that a person will hear that there is going to be an election.

♦6. John makes it to work on time one Monday out of four. On other work days his behaviour is as follows: If he is late one day, he is twice as likely to come to work on time the next day as to be late. If he is on time one day, he is as likely to be late as not the next day. Find the probability of his being late and that of his being on time Wednesdays.

7. Suppose you have 1¢ and match coins with a friend. At each match you either win or lose 1¢ with equal probability. If you go broke or ever get 4¢, you quit. Assume your friend never quits. If the states are 0, 1, 2, 3, and 4 representing your wealth, show that the corresponding transition matrix P is not regular. Find the probability that you will go broke after 3 matches.

8. A mouse is put into a maze of compartments, as in the diagram. Assume that he always leaves any compartment he enters and that he is equally likely to take any tunnel entry.
   ♦(a) If he starts in compartment 1, find the probability that he is in compartment 4 after 3 moves.
   ♦(b) Find the compartment in which he spends most of his time if he is left for a long time.



9. If a stochastic matrix has a 1 on its main diagonal, show that it cannot be regular. Assume it is not 1×1.

10. If S_m is the stage-m state vector for a Markov chain, show that $S_{m+k} = P^k S_m$ holds for all $m \geq 1$ and $k \geq 1$ (where P is the transition matrix).

11. A stochastic matrix is **doubly stochastic** if all the row sums also equal 1. Find the steady-state vector for a doubly stochastic matrix.

♦12. Consider the 2×2 stochastic matrix
$P = \begin{bmatrix} 1-p & q \\ p & 1-q \end{bmatrix}$, where $0 < p < 1$ and $0 < q < 1$.
   (a) Show that $\frac{1}{p+q}\begin{bmatrix} q \\ p \end{bmatrix}$ is the steady-state vector for P.
   (b) Show that P^m converges to the matrix $\frac{1}{p+q}\begin{bmatrix} q & q \\ p & p \end{bmatrix}$ by first verifying inductively that
   $$P^m = \frac{1}{p+q}\begin{bmatrix} q & q \\ p & p \end{bmatrix} + \frac{(1-p-q)^m}{p+q}\begin{bmatrix} p & -q \\ -p & q \end{bmatrix}$$
   for $m = 1, 2, \ldots$. (It can be shown that the sequence of powers $P, P^2, P^3, \ldots$ of any regular transition matrix converges to the matrix each of whose columns equals the steady-state vector for P.)

Supplementary Exercises for Chapter 2

1. Solve for the matrix X if: (a) $PXQ = R$; (b) $XP = S$; where
$$P = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 3 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix},$$
$$R = \begin{bmatrix} -1 & 1 & -4 \\ -4 & 0 & -6 \\ 6 & 6 & -6 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 6 \\ 3 & 1 \end{bmatrix}$$

2. Consider $p(X) = X^3 - 5X^2 + 11X - 4I$.
   (a) If $p(A) = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}$, compute $p(A^T)$.
   ♦(b) If $p(U) = 0$ where U is $n \times n$, find U^{-1} in terms of U.

3. Show that, if a (possibly nonhomogeneous) system of equations is consistent and has more variables than equations, then it must have infinitely many solutions. [*Hint:* Use Theorem 2 §2.2 and Theorem 1 §1.3.]

4. Assume that a system $AX = B$ of linear equations has at least two distinct solutions Y and Z.
   (a) Show that $X_k = Y + k(Y - Z)$ is a solution for every k.
   ♦(b) Show that $X_k = X_m$ implies $k = m$. [*Hint:* See Example 7 §2.1.]
   (c) Deduce that $AX = B$ has infinitely many solutions.

5. (a) Let A be a 3×3 matrix with all entries on and below the main diagonal zero. Show that $A^3 = 0$.
   (b) Generalize to the $n \times n$ case and prove your answer.

6. Let I_{pq} denote the $n \times n$ matrix with (p, q)-entry equal to 1 and all other entries 0. Show that:

(a) $I_n = I_{11} + I_{22} + \cdots + I_{nn}$.

(b) $I_{pq}I_{rs} = \begin{cases} I_{ps} & \text{if } q = r \\ 0 & \text{if } q \neq r. \end{cases}$

(c) If $A = [a_{ij}]$ is $n \times n$, then $A = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}I_{ij}$.

♦(d) If $A = [a_{ij}]$, then $I_{pq}AI_{rs} = a_{qr}I_{ps}$ for all p, q, r, and s.

7. A matrix of the form aI_n, where a is a number, is called an $n \times n$ **scalar matrix**.

(a) Show that each $n \times n$ scalar matrix commutes with every $n \times n$ matrix.

♦(b) Show that A is a scalar matrix if it commutes with every $n \times n$ matrix. [*Hint:* See part (d) of Exercise 6.]

8. Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where A, B, C, and D are all $n \times n$ and each commutes with all the others. If $M^2 = 0$, show that $(A + D)^3 = 0$. [*Hint:* First show that $A^2 = -BC = D^2$ and that $B(A + D) = 0 = C(A + D)$.]

9. If A is 2×2, show that $A^{-1} = A^T$ if and only if

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \text{ for some } \theta$$

$$\text{or } A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \text{ for some } \theta.$$

[*Hint:* If $a^2 + b^2 = 1$, then $a = \cos\theta$, $b = \sin\theta$ for some θ. Use $\cos(\theta - \varphi) = \cos\theta\cos\varphi + \sin\theta\sin\varphi$.]