Soln.

Solution of test paper of iit-jee pattern- Paper-01

29.

$$P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Here $PP^T = I \quad \therefore P^T = P^{-1}$

Now, given that $Q = PAP^T$

$$\Rightarrow P^TQ = P^TPAP^T$$

$$= AP^T \qquad \left[\because P^TP = I,\ IA = A\right]$$

$$\Rightarrow P^TQ^{2005}P = AP^TQ^{2004}P$$

$$= AP^TQ^{2003}PA$$

$$(\because Q = PAP^T \Rightarrow QP = PA)$$

$$= AP^TQ^{2002}PA^2 = AP^TPA^{2004}$$

$$= AIA^{2004} = A^{2005} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$$

30. $\because f(x) = \sin^{-1}\{4-(x-7)^3\}^{\frac{1}{5}}$

Let $f(x) = y$

$$\therefore y = \sin^{-1}\{4-(x-7)^3\}^{\frac{1}{5}}$$

$$\Rightarrow \sin y = \{4-(x-7)^3\}^{\frac{1}{5}}$$

$$\Rightarrow \sin^5 y = 4 - (x-7)^3$$

$$\Rightarrow (x-7)^3 = 4 - \sin^5 y$$

$$\Rightarrow x - 7 = \{4 - \sin^5 y\}^{\frac{1}{3}}$$

$$\Rightarrow x = 7 + \{4 - \sin^5 y\}^{\frac{1}{3}}$$

$$\Rightarrow f^{-1}(x) = \left[7 + \{4 - \sin^5 y\}^{\frac{1}{3}}\right]$$

31. $A = \{1, 2, 3\}$ R is Relation from A to A

$\therefore$ for equivalence relation it must be reflexive, symmetric and transitive

$R = \{(1, 2)\}$

for reflexive it must have $\{(1,1)\ (2,2)\ (3,3)\}$ elements in R:

for symmetric $(1,2)\ (2,1)\ (1,3)\ (3,1)\ (2,3)\ (3,1)$ must be elements in R

then transitive relation will also hold good in R.

$\therefore$ New minimum no. of elements in R

$(1,1)\ (2,2)\ (3,3)\ (2,1)$

$\therefore$ No. of minimum elements $= 4$

32. $f(x) = \sin^2 x + \sin^2\left(x+\frac{\pi}{3}\right) + \cos x \cdot \cos\left(x+\frac{\pi}{3}\right)$

$g\left(\frac{5}{4}\right) = 1$

$$f(x) = 1 - \cos^2 x + \sin^2\left(x+\frac{\pi}{3}\right) + \cos x \cos\left(x+\frac{\pi}{3}\right)$$

$$= 1 - \left[\cos^2 x - \sin^2\left(x+\frac{\pi}{3}\right)\right] + \cos x \cos\left(x+\frac{\pi}{3}\right)$$

$$= 1 - \cos\left(x+x+\frac{\pi}{3}\right)\cos\left(x-x-\frac{\pi}{3}\right) + \cos x \cos\left(x+\frac{\pi}{3}\right)$$

$$= 1 - \cos\left(2x+\frac{\pi}{3}\right)\left(\frac{1}{2}\right) + \frac{1}{2}\, 2\cos\left(x+\frac{\pi}{3}\right)\cos x$$

$$= 1 - \frac{1}{2}\cos\left(2x+\frac{\pi}{3}\right) + \frac{1}{2}\left[\cos\left(2x+\frac{\pi}{3}\right) + \cos\frac{\pi}{3}\right]$$

$$= 1 + \frac{1}{2}\cos\frac{\pi}{3} = 1 + \frac{1}{2} \times \frac{1}{2} = 1 + \frac{1}{4} = \frac{5}{4}$$

$f(x) = \frac{5}{4}$ i.e constant function

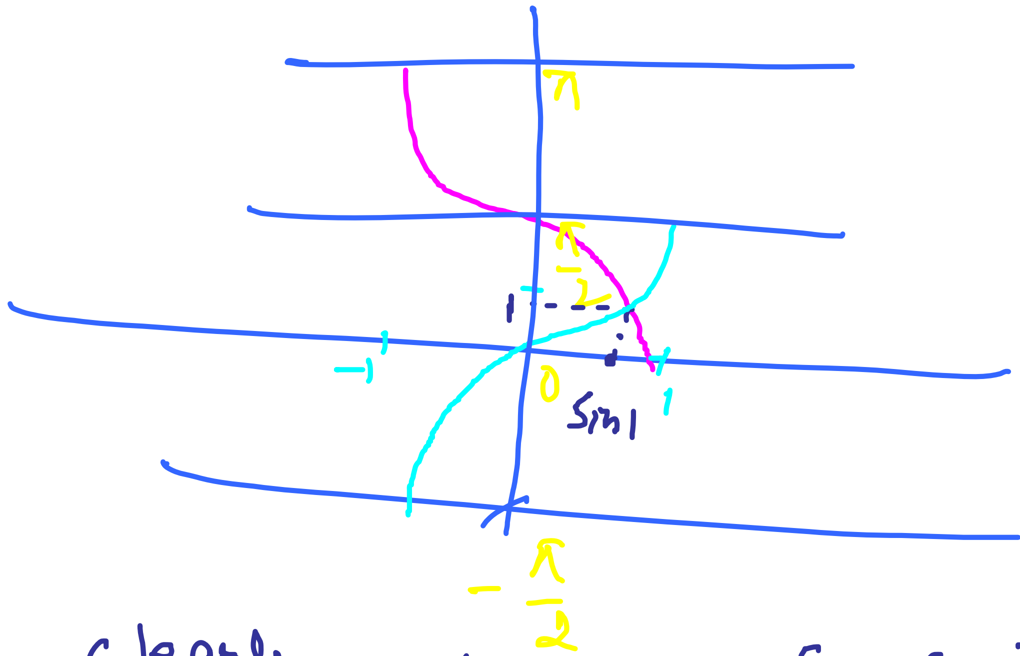
$\therefore$ $g \circ f(x) = g\left(\frac{5}{4}\right) = 1$ given.

Then $(g \circ f)x$ is a constant function.

33. $[\sin^{-1} x] > [\cos^{-1} x]$ where [ ] denotes greatest Integer function

This type of question can be solve by graphical method easily.

Graphs of $\sin^{-1}x$ and $\cos^{-1}x$ are as follows



clearly, when $x \in [0, \sin 1)$

$[\sin^{-1}x] = 0$ while $[\cos^{-1}x] \geq 1$

But when $x \in [\sin 1, 1]$

$[\sin^{-1}x] = 1$ while $[\cos^{-1}x] = 0$

$\therefore$ clearly, $[\sin^{-1}x] > [\cos^{-1}x]$ in $[\sin 1, 1]$

34. $xe^{xy} = y + \sin^2 x$, then $\frac{dy}{dx}$ at $x=0$

Differentiating both the sides we get

$$e^{xy} + x\,e^{xy}\left[y + x\frac{dy}{dx}\right] = \frac{dy}{dx} + \sin 2x$$

when $x=0$, $y=0$ in given equation

$\therefore\ e^0 + 0 = \frac{dy}{dx} + 0 \Rightarrow \frac{dy}{dx} = 1$

35. $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ and $A^2 - 4A = kI_3$.

Then value of k is

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

when $A^2 - 4A = kI_3$.

$$\Rightarrow \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$$

$\Rightarrow k = 5$

36. Range of $y = \dfrac{x^2 - 3x + 2}{x^2 + x - 6}$

$\Rightarrow yx^2 + xy - 6y = x^2 - 3x + 2$

$\Rightarrow (y-1)x^2 + (y+3)x - 2(3y+1) = 0$

for real x, $D \geq 0$

$(y+3)^2 - 4(y-1)[-2(3y+1)] \geq 0$

$\Rightarrow y^2 + 6y + 9 + 8(3y^2 - 2y - 1) \geq 0$

$\Rightarrow 25y^2 - 10y + 1 \geq 0 \Rightarrow (5y-1)^2 \geq 0$

it is true for any real value of y.

Now,

$$x = \frac{-(y+3) \pm \sqrt{(5y-1)^2}}{2(y-1)}$$

$$= \frac{-y-3 \pm (5y-1)}{2(y-1)}$$

Here $y \neq 1$, when $5y-1=0$, then maximum and minimum cannot hold

$\therefore y \neq \frac{1}{5}$

$\therefore$ Range $(-\infty, \infty) - \{\frac{1}{5}, 1\}$

Section II

37. $A = [a_{ij}]_{n\times n}$ and $|A| = 3$ given

for n=3, $|A^{-1}| = \left|\frac{1}{|A|}(\text{adj } A)\right|$

$$= \left|\frac{1}{3}(\text{adj } A)\right| = \frac{1}{3}|\text{adj } A|$$

$$= \frac{1}{3}|A|^2 = \left(\frac{1}{3}\right)^3 \times 9 = \frac{1}{3}$$

[$\because$ matrix is of order 3, therefore, if we multiply by k the value of determinant of that matrix will be multiply by k^3.

and $|\operatorname{Adj} A| = |A|^{n-1}$]

For $n=2$, $|\operatorname{Adj} A| = |A|^{2-1} = |A|$

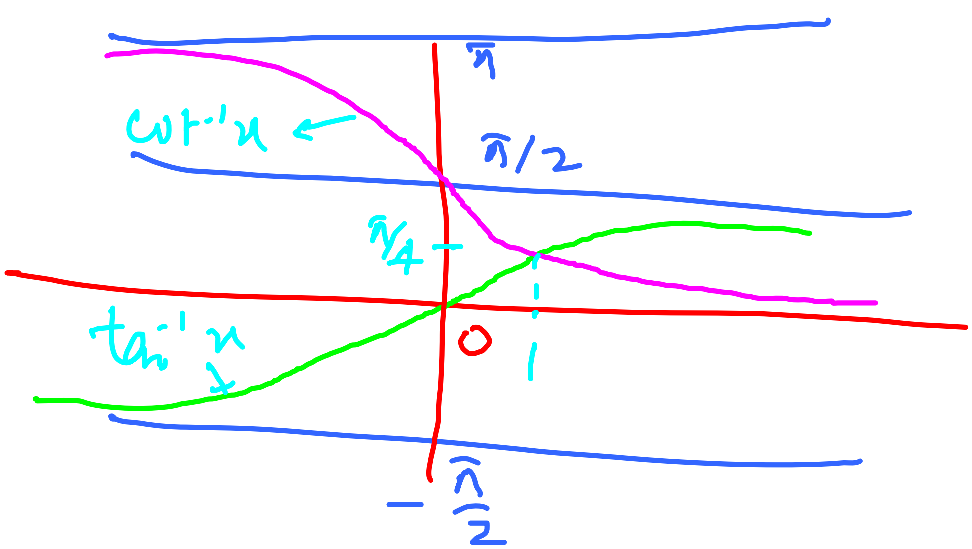
38. $f(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$f\left(\frac{\pi}{3}\right) f\left(\frac{\pi}{6}\right) = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = f\left(\frac{\pi}{2}\right)$$

Similarly, $f^4\left(\frac{\pi}{3}\right) = f\left(\frac{2\pi}{3}\right)$

39. $\tan^{-1}x > \cot^{-1}x$



Out of the given option, it is clear from the graph

$\tan^{-1}x > \cot^{-1}x$ hold in $[1, \infty)$

$\therefore$ also in $[2, \infty)$

40. $\cos^{-1}(x^2 - 6x + 8.5) = \pi/3$.

$$x^2 - 6x + \frac{17}{2} = \frac{1}{2}$$

$$\Rightarrow 2x^2 - 12x + 17 = 1$$

$$\Rightarrow 2x^2 - 12x + 16 = 0$$

$$\Rightarrow x^2 - 6x + 8 = 0 \Rightarrow (x-2)(x-4) = 0$$

$$\Rightarrow x = 2, 4$$

Also, $-1 \leq x^2 - 6x + 8.5 \leq 1$

$$-1 \leq x^2 - 6x + 9 - 9 + 8.5 \leq 1$$

$$-1 \leq (x-3)^2 - \frac{1}{2} \leq 1$$

$$-\frac{3}{2} \leq (x-3)^2 \leq \frac{3}{2}$$

$$\Rightarrow (x-3)^2 - \sqrt{\frac{3}{2}} \leq 0$$

$$\left(x - 3 + \sqrt{\frac{3}{2}}\right)\left(x - 3 - \frac{\sqrt{3}}{2}\right) \leq 0$$

$$\Rightarrow 3 - \frac{\sqrt{3}}{2} \leq x \leq 3 + \frac{\sqrt{3}}{2}$$

∵ Since 2 and 4 lies in above Interval ∴ Hence 2, 4 are Integral Sol^n

∴ Total two Solution

41 $t_k = \tan^{-1}\left(\frac{2}{k^2}\right)$

$= \tan^{-1}\left[\frac{(k+1)-(k-1)}{1+(k-1)(k+1)}\right]$

$= \tan^{-1}(k+1) - \tan^{-1}(k-1)$

$$\therefore \sum_{k=1}^{n} t_k = \left(\cancel{\tan^{-1} 2} - \tan^{-1} 1\right) + \left(\cancel{\tan^{-1} 3} - \cancel{\tan^{-1} 2}\right) + \cancel{\cdots} \cancel{\cdots} + \left[\tan^{-1}(n+1) - \cancel{\tan^{-1}(n-1)}\right]$$

$= \tan^{-1}(n+1) - \tan^{-1} 1$

$= \tan^{-1}\frac{n+1-1}{1+(n+1)} = \tan^{-1}\frac{n}{n+2}$ (correct)

When $n \to \infty$ $\sum_{k=1}^{n} t_k = \tan^{-1}\infty - \tan^{-1} 1$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ (correct)

Comprehension Type Section III

$f(x) = e^x$ and $g(x) = 3x - 2$

42. $fog(x) = f(3x-2) = e^{3x-2} = y$ (say)

$\therefore 3x - 2 = \log_e y \Rightarrow x = \frac{1}{3}[\log_e y + 2]$

$\therefore f^{-1}(y) = \frac{1}{3}[\log_e y + 2]$

$\therefore (fog)^{-1} = \frac{1}{3}[\log_e x + 2]$

$\therefore$ domain of $(fog)^{-1}$ is given by

$$x > 0$$

which is true for all $x \in R$ i.e $(0, \infty)$

43. $gof(x) = g(e^x) = 3e^x - 2 = y$ (say)

$\therefore 3e^x = y + 2 \Rightarrow x = \log_e\left(\frac{y+2}{3}\right)$

$f^{-1}(y) = \log_e\left(\frac{y+2}{3}\right)$

$\Rightarrow (gof)^{-1} = \log_e\left(\frac{x+2}{3}\right)$

44 to 46.

Let $\begin{vmatrix} x^4 + x^2 & x^3 - 1 & 2x^2 - x + 1 \\ 3 & -1 & 2 \\ 4 & 1 & 3 \end{vmatrix} = px^4 + qx^3 + rx^2 + sx + t$

44. Put $x=0$, $\begin{vmatrix} 0 & -1 & 1 \\ 3 & -1 & 2 \\ 4 & 1 & 3 \end{vmatrix} = t$

$$\therefore t = 1(9-8) + 1(3+4) = 8 \quad \text{— (i)}$$

Put $x=1$, $\begin{vmatrix} 2 & 0 & 2 \\ 3 & -1 & 2 \\ 4 & 1 & 3 \end{vmatrix} = p+q+r+s+t$

$$\Rightarrow 2(-3-2) + 2(3+4) = p+q+r+s+8$$

$$\Rightarrow -10+14 = p+q+r+s+8$$

$$\Rightarrow p+q+r+s = -4 \quad \text{—— (ii)}$$

Put $x=-1$, $\begin{vmatrix} 2 & -2 & 4 \\ 3 & -1 & 2 \\ 4 & 1 & 3 \end{vmatrix} = p-q+r-s+t$

$$\Rightarrow 2(-3-2) + 2(9-8) + 4(3+4) = p-q+r-s+8$$

$$\Rightarrow -10+2+28-8 = p-q+r-s$$

$$\Rightarrow p-q+r-s = 12 \quad \text{— (iii)}$$

Adding (ii) and (iii) we get

$$2(p+r) = 8 \Rightarrow \underline{p+r=4}$$

$\therefore\ p+r+t = 4+8 = 12$

from (iii) $\quad q+s = p+r-12 = -8$

Compairising heighest coefficient both the sides $\quad p=-5$

from (ii) $\quad q=-1 \quad \therefore\ s = -8-q$

$= -8+1 = -7$

$s=-7$

44. $s=-7$

45. $p+r+t = 12$

46. $p+r = 4$

Section IV

47. $\sin\left(3\cos^{-1}x + \sin^{-1}x\right)$ at $x = \frac{1}{5} = \frac{2-k^2}{k^2}$

$= \sin\left(2\cos^{-1}x + \frac{\pi}{2}\right) = \cos\left(2\cos^{-1}x\right)$

$= \cos\left(\cos^{-1}(2x^2-1)\right)$

when $x = \frac{1}{5}$

$= \cos\left(\cos^{-1}\frac{-23}{25}\right) = -\frac{23}{25} \quad \left[\because -1 \le \frac{-23}{25} \le 1\right]$

$\therefore\ -\frac{23}{25} = \frac{2-k^2}{k^2} \Rightarrow \frac{2}{k^2} - 1 = -\frac{23}{25}$

$\Rightarrow \frac{2}{k^2} = \frac{2}{25} \Rightarrow k=5$

48. Let $f(x) = \begin{cases} 1+[x] & , x<-2 \\ |x| & , x \geq -2 \end{cases}$

where $[x]$ is greatest integer function

Then $f(-2.6) = 1+[-2.6] = 1-3 = -2$

$\therefore f(f(2.6)) = f(-2) = |-2| = \underline{2}$

49. Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

$|A| = 2(3-0)-1(5-0) = 1$

$A^{-1} = \begin{bmatrix} 3 & -1 & q \\ -15 & 6 & -5 \\ p & -2 & 2 \end{bmatrix}$

$\therefore q =$ cofactor of $C_{31} = (-1)^{3+1}(0+1) = 1$

$p =$ cofactor of $C_{13} = (-1)^{1+3}(5-0) = 5$

$\therefore p+q = \underline{6}$

50. $\sin^{-1}(ax)+\sin^{-1}y = \frac{\pi}{2}$, $a>0$

$\Rightarrow \sin^{-1}ax = \frac{\pi}{2} - \sin^{-1}y$

$\Rightarrow \sin^{-1}ax = \cos^{-1}y$

$\Rightarrow \sin^{-1}ax = \sin^{-1}\sqrt{1-y^2}$

$ax = \sqrt{1-y^2} \Rightarrow a^2x^2+y^2 = 1$

$\because$ given that $x^2+y^2=1 \Rightarrow a=1$

51. Since given matrix A is skew symmetric matrix. Hence

$\det(A) = 0$

$\therefore \sqrt{64-\det A} = \sqrt{64-0} = 8$

52. $\because f(x)$ and $g(x)$ are inverse of each other $\Rightarrow g(f(x)) = x$

$g'(f(x))\,f'(x) = 1$

when $x=1$, $g'(f(1))\,f'(1) = 1$

$g'(5)\times 4 = 1 \Rightarrow \frac{1}{g'(5)} = 4$

53. $\tan^{-1}(\tan 3) = \tan^2 x$.

$\tan^{-1}\tan(3-\pi) = \tan^2 x$

$\because 3 \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$3-\pi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$3-\pi = \tan^2 x$.

$\tan(3-\pi) = -\tan(\pi-3)$

$= \tan 3$

$\tan^2 x > 0$, But $3-\pi < 0$

$\therefore$ Equation can not hold for any value of x.

$\therefore$ No Solution. No. of solution $\underline{= 0}$

54. $\lambda, \mu, \nu > 10$

$\mu = 10x+1$ and $\nu = 10y+0 = 10y$

where x and y are integers in $[1,9]$

$$\therefore \Delta = \begin{vmatrix} \lambda & 4 & 1 \\ \mu & 0 & 1 \\ \nu & 1 & 0 \end{vmatrix} = \lambda(-1) + \mu(1) + \nu(4)$$

$$= -\lambda + \mu + 4\nu$$

$\because$ $\Delta + 1$ is divisible by 10

$\therefore$ $1 - \lambda + \mu + 4\nu = 10k \quad (k \in I)$

$1 - \lambda + 10x + 1 + 40y = 10k.$

$\lambda = 10x + 40y + 2 - 10k$

$= 10(x-k) + 40y + 2$

$\therefore$ Unit place of λ is $\underline{2.}$

55 For Non trivial soln. $|A| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$R_1 \to R_1 + R_2 + R_3$

$$\begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (6-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$\lambda = 6$

56. $f(1) = 4$