Solution of test paper -02

Section I

20.
$$\begin{vmatrix} x^k & x^{k+2} & x^{k+3} \\ y^k & y^{k+2} & y^{k+3} \\ z^k & z^{k+2} & z^{k+3} \end{vmatrix} = (x-y)(y-z)(z-x)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

$$= \frac{(x-y)(y-z)(z-x)(xy+yz+zx)}{xyz}$$

i.e xyz Comes in denom.

Put $k=-1$

Then
$$\begin{vmatrix} \frac{1}{x} & x & x^2 \\ \frac{1}{y} & y & y^2 \\ \frac{1}{z} & z & z^2 \end{vmatrix} = \frac{1}{xyz}\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix}$$

$$= \frac{1}{xyz}(x-y)(y-z)(z-x)(xy+yz+zx)$$

(Standard result)

$\therefore \underline{k=-1}$

21. $A = BX$, $A = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$$\Rightarrow X = B^{-1}A = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 4 \\ 3 & -5 \end{bmatrix}$$

22. $x = \cos^{-1}\cos 4 = \cos^{-1}\cos(2\pi - 4) = 2\pi - 4$

$y = \sin^{-1}\sin 3 = \sin^{-1}\sin(\pi - 3) = \pi - 3.$

$x + y = 3\pi - 7$

$\tan(x+y) = \tan(3\pi - 7) = -\underline{\tan 7}$

23. $x^3 + bx^2 + cx + 1 = 0 \quad (b < c)$

$\because \alpha$ is the only real root of equation

Hence α must be negative, i.e $\alpha < 0$

[$\because$ any eq^n of odd degree with real coefficient has at least one real root opposite to the sign of its constant term)

$\therefore \tan^{-1}\alpha + \tan^{-1}\frac{1}{\alpha} = \tan^{-1}\alpha + \cot^{-1}\alpha = -\frac{\pi}{2}$

24. $f(1) \leq f(2)$, $f(3) \geq f(4)$ and $f(5) = 5$

$\because$ $f(x)$ is a linear function

$\therefore$ let $f(x) = ax + b$.

$f(1) \leq f(2) \Rightarrow a + b \leq 2a + b.$

$\Rightarrow a \geq 0$ — (I)

Also, $f(3) \geq f(4)$

$\Rightarrow 3a + b \geq 4a + b \Rightarrow a \leq 0$ — (II)

from (I) and (II) $a = 0$

$f(5) = 5 \Rightarrow 5a+b = 5 \Rightarrow b = 5$

$\therefore f(x) = ax+b$
$= 0 \cdot x + 5 \Rightarrow f(x) = 5$ Constant fun.
$\underline{f(0) = 5}$

25. $f(x) = \sin\left(\log\left(x+\sqrt{x^2+1}\right)\right)$

$f(-x) = \sin\left(\log\left(\sqrt{x^2+1} - x\right)\right)$

$= \sin \log\left(\dfrac{\sqrt{x^2+1} - x}{\sqrt{x^2+1}+x} \times \sqrt{x^2+1}+x\right)$

$= \sin\left(\log\left(\dfrac{\cancel{x^2}+1-\cancel{x^2}}{\sqrt{x^2+1}+x}\right)\right)$

$= \sin\left(\log\left(\sqrt{x^2+1}+x\right)^{-1}\right)$

$= \sin\left(-\log\left(\sqrt{x^2+1}+x\right)\right)$

$= -\sin\left(\log\left(\sqrt{x^2+1}+x\right)\right) = -f(x)$

$\therefore$ $f(x)$ is odd function and statement I is true.

<u>Statement II</u> Period of $f(x) = 2\cos \frac{1}{3}(x-\pi)$ is 4π

Period of $\cos x$ is 2π

$\therefore$ Period of $2\cos\left(\frac{1}{3}x - \frac{\pi}{3}\right)$ is $\dfrac{2\pi}{\frac{1}{3}} = 6\pi$

$\therefore$ Statement II is false.

Only sequence out of given option is T F F

(Statement III is false because it is even function whose graph will be symmetrical about y axis)

Section II

26. Period of $f(x) = \cos nx \sin \frac{5x}{n}$ is 3π

period of $\cos nx$ is $\frac{2\pi}{n}$

Period of $\sin \frac{5x}{n}$ is $\frac{2\pi}{\frac{5}{n}} = \frac{2n\pi}{5}$

LCM of $\frac{2\pi}{n}$ and $\frac{2n\pi}{5}$ is 3π

Clearly, n is multiple of 5

when $n=5$, LCM of $\frac{2\pi}{n}$ and $\frac{2n\pi}{5}$ is 2π

$n=10$, LCM of $\frac{2\pi}{n}$ and $\frac{2n\pi}{5}$ is 4π

$n=15$, LCM of $\frac{2\pi}{n}$ and $\frac{2n\pi}{5}$ is 3π

$f(x) = \cos 15x \sin \frac{x}{3}$

$f(x+3\pi) = \cos 15(x+3\pi) \sin \frac{\cancel{5}(x+3\pi)}{\cancel{15}\,3}$

$= \cos(45\pi + 15x) \sin\left(\pi + \frac{x}{3}\right)$

$= (-\cos 15x)(-\sin x) = \cos 15x \sin x.$

$= f(x)$

Then, $\dfrac{105}{x^2 - 14x} = \dfrac{105}{225 - 210} = \dfrac{105}{15} = 7$ Ans.

(27) $\Delta = \begin{vmatrix} f(x) & f(x) + f(\frac{1}{x}) \\ 1 & f(\frac{1}{x}) \end{vmatrix} = 0, \ x \neq 0$

$\Rightarrow f(x) + f(\frac{1}{x}) = f(x)\, f(\frac{1}{x})$ — (1)

and $f(x) = a + bx^n$.

$f(\frac{1}{x}) = a + \dfrac{b}{x^n}$.

from (1) $a + bx^n + a + \dfrac{b}{x^n} = (a + bx^n)(a + \dfrac{b}{x^n})$

$\Rightarrow 2a + b(x^n + \frac{1}{x^n}) = a^2 + ab(x^n + \frac{1}{x^n}) + b^2$

$\Rightarrow 2a + b(x^n + \frac{1}{x^n}) = a^2 + b^2 + ab(x^n + \frac{1}{x^n})$

$\Rightarrow ab = b \Rightarrow b(a-1) = 0 \Rightarrow a = 1, \ b \neq 0$

and $a^2 + b^2 = 2a$

when $a = 1$, $b^2 = 1 \Rightarrow b = \pm 1$

$f(x) = 1 \pm x^n$.

But $f(2) = 17 \Rightarrow f(2) = 1 + 2^4$

$\therefore f(x) = x^4+1$ on comparing

$f(5) = 5^4 + 1 = 625 + 1 = 625$

$$\therefore \left[\frac{f(5)}{500}\right] = \left[\frac{625}{500}\right] = \left[\frac{5}{4}\right] = 1$$

28. $\tan^{-1}(x+1) + \tan^{-1}x + \tan^{-1}(x-1) = \tan^{-1}3$

$\Rightarrow \tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}3 - \tan^{-1}x.$

$$\Rightarrow \tan^{-1}\frac{x+1+x-1}{1-(x+1)(x-1)} = \tan^{-1}\frac{3-x}{1+3x}$$

$$\Rightarrow \frac{2x}{2-x^2} = \frac{3-x}{1+3x}$$

$\Rightarrow 2x + 6x^2 = 6 - 2x - 3x^2 + x^3.$

$\Rightarrow x^3 - 9x^2 - 4x + 6 = 0$

$\Rightarrow$ which satisfy by $x = -1$

$\therefore x^4 + 1 = (-1)^4 + 1 = 2$

29. $f(x) = x^2+1 \Rightarrow y = x^2+1$ say.

$\Rightarrow x^2 = y - 1$

$x = \sqrt{y-1}$

$\therefore f^{-1}(x) = \sqrt{x-1}$

Now $f^{-1}(17) = \sqrt{17-1} = \sqrt{16} = 4$

$f^{-1}(-3) = \sqrt{-3-1} = \sqrt{-4}$ do not exists

as $f: R \to R$ $\therefore f^{-1}(-3) = \phi$

$\therefore f^{-1}(17) \cup f^{-1}(-3) = \{4, \phi\}$

$\therefore$ No. of elements is 2.

30. $\because$ f is even function on interval $(-5, 5)$

and $f(x) = f\left(\frac{x+1}{x+2}\right)$

$[\because f(-x) = f(x)]$

$\therefore \frac{x+1}{x+2} = \pm x$

when $\frac{x+1}{x+2} = -x$ $\Rightarrow x+1 = -x^2 - 2x$

$\Rightarrow x^2 + 3x + 1 = 0$

$\Rightarrow x = \frac{-3 \pm \sqrt{5}}{2} \in (-5, 5)$

when $\frac{x+1}{x+2} = x \Rightarrow x+1 = x^2 + 2x$

$\Rightarrow x^2 + x - 1 = 0$

$\Rightarrow x = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$

$= \frac{-1 \pm \sqrt{5}}{2} \in (-5, 5)$

$\therefore$ Total no. of solution in $(-5, 5)$ is 4

Section III

$\sin^{-1}x \in \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$, $\cos^{-1}x \in [2\pi, 3\pi]$

$x \in [-1, 1]$, $\tan^{-1}x \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$

$x \in R$

31. $\sin^{-1}x + \cos^{-1}y = \frac{11\pi}{2}$

which will satisfy only when $\sin^{-1}x = \frac{5\pi}{2}$

$\cos^{-1}y = 3\pi$

Hence, there is only one solution

32. $\sin^{-1}\cos\left(\frac{5\pi}{4}\right) = \sin^{-1}\left(\cos\left(2\pi - \frac{3\pi}{4}\right)\right)$

$= \sin^{-1}\cos\frac{3\pi}{4} = \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

$= \sin^{-1}\sin\left(2\pi - \frac{\pi}{4}\right)$

$= \sin^{-1}\sin\frac{7\pi}{4} = \frac{7\pi}{4}$

$\therefore \sin^{-1}x \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$

33. $2\sin^{-1}x = \sin^{-1}2x\sqrt{1-x^2}$

put $x = \sin\theta$

RHS $= \sin^{-1}(2\sin\theta\cos\theta)$

$= \sin^{-1}\sin 2\theta$

$= 2\theta$

$= 2\sin^{-1}x.$

if $-\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2}$

$-\pi/4 \le \theta \le \pi/4$

$$\Rightarrow \quad -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$$

34 to 36.

$$f(x) = \begin{vmatrix} x & -16 \\ 9 & x-\lambda \end{vmatrix}$$

$$= x(x-\lambda) + 144$$

$$f(x) = 0 \Rightarrow x^2 - \lambda x + 144 = 0$$

$$\Rightarrow x = \frac{\lambda \pm \sqrt{\lambda^2 - 576}}{2}$$

for real value of x,

$$\lambda^2 - 576 \geq 0$$

$$\Rightarrow (\lambda)^2 - (24)^2 \geq 0$$

$\Rightarrow$ 

i.e $\lambda \leq -24$ or $\lambda \geq 24$

Also, for integer $\lambda^2 - 576$ must be a perfect square and must be multiple of 4 so that after square root it would be divisible by 2. and λ should be integral multiple of 2

$\lambda = \pm 24$, $\quad x = \frac{\pm 24 \pm 0}{2}$ gives integral value

$\lambda = \pm 25$,

$\lambda = \pm 26$, $\quad x = \frac{\pm 26 \pm 10}{2}$ "

$\lambda = \pm 30$, $x = \dfrac{\pm 30 \pm 18}{2}$ ''

$\lambda = \pm 40$, $x = \dfrac{\pm 40 \pm 32}{2}$ '' etc.

$\lambda = \pm 145$, $x = \pm 145$

34. The no. of values of λ for which root of $f(x)=0$ integers $= 16$

35 maximum value of λ is 145

36 Sum of all values of $\lambda = 0$ (as they are equally positive and negative values).

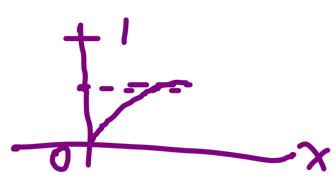
Section iv

Column match

(A) $f : \left[0, \frac{\pi}{3}\right] \to [0, 1]$

$$f(x) = \sqrt{\sin x}$$

Graph of $f(x) =$



Range of $f(x)$ is $\left[0, \dfrac{3^{1/4}}{2^{1/2}}\right] \subseteq [0, 1]$

Hence, $f(x)$ is one to one into function

(B) $f : (1, \infty) \to (1, \infty)$

$$f(x) = \frac{x+3}{x-1} \Rightarrow f'(x) = \frac{-4}{(x-1)^2} < 0$$

Let $y = \frac{x+3}{x-1}$ $\Rightarrow$ $xy - y = x+3$.

$\Rightarrow x(y-1) = 3+y$

But $x \in (1, \infty)$ $\Rightarrow x = \frac{3+y}{y-1}$

when $x > 1$

$$\frac{3+y}{y-1} > 1 \Rightarrow \frac{y+3}{y-1} - 1 > 0$$

$$\Rightarrow \frac{\cancel{y}+3-\cancel{y}+1}{y-1} > 0 \Rightarrow \frac{4}{y-1} > 0$$

$y > 1$

Range $(1, \infty)$

Codomain $(1, \infty)$ $\therefore$ Range = codomain

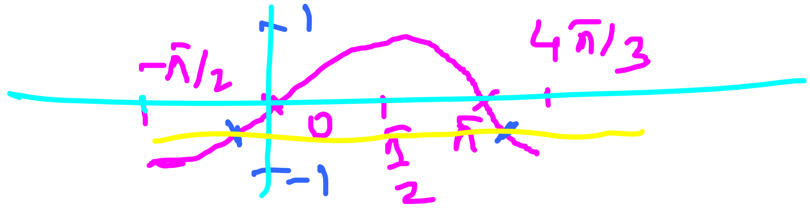
$\therefore$ $f(x)$ is one to one onto function

$\therefore$ $f(x)$ is bijective

(c) $f: \left[-\frac{\pi}{2}, \frac{4\pi}{3}\right] \rightarrow [-1, 1]$

$f(x) = \sin x$

Graphs



Clearly, $f(x) = \sin x$ is many to one function in $\left[-\frac{\pi}{2}, \frac{4\pi}{3}\right]$

Range $[-1, 1]$ Hence onto function

(D) $f : (2, \infty) \to [8, \infty)$

$$f(x) = \frac{x^2}{x-2} = \frac{x^2 - 4 + 4}{x-2}$$

$$= x + 2 + \frac{4}{x-2}$$

$$f'(x) = 1 - \frac{4}{(x-2)^2}$$

$f'(x) = 0,$ $\quad (x-2)^2 = 4$

$$x - 2 = \pm 2$$

$$x = 0, 4$$

Clearly. $f(x)$ is neither strictly increasing nor decreasing. Hence many to one functi

Now, let $y = \dfrac{x^2}{x-2} \Rightarrow xy - 2y = x^2$

$$\Rightarrow x^2 - xy + 2y = 0$$

$$x = \frac{y \pm \sqrt{y^2 - 8y}}{2}$$

for real x $\quad y^2 - 8y \geq 0$

$$y(y-8) \geq 0 \Rightarrow y \leq 0 \text{ or } y \geq 8$$

But $x \in (2, 8)$

$\therefore$ when $y < 0$, it not satisfy in domain.

Range $[8, \infty)$ = codomain.

$\therefore$ onto function

38.

$a_k = {}^nC_k$

$$\therefore A_k = \begin{bmatrix} {}^nC_{k-1} & 0 \\ 0 & {}^nC_k \end{bmatrix}$$

$$B = \sum_{k=1}^{n-1} A_k A_{k+1} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$\sum_{k=1}^{n-1} \begin{bmatrix} {}^nC_{k-1} & 0 \\ 0 & {}^nC_k \end{bmatrix} \begin{bmatrix} {}^nC_k & 0 \\ 0 & {}^nC_{k+1} \end{bmatrix}$$

$$\sum_{k=1}^{n-1} \begin{bmatrix} {}^nC_{k-1}\,{}^nC_k & 0 \\ 0 & {}^nC_k\,{}^nC_{k+1} \end{bmatrix}$$

$$= \begin{bmatrix} {}^nC_0\,{}^nC_1 + {}^nC_1\,{}^nC_2 + \dots + {}^nC_{n-1}\,{}^nC_n & 0 \\ 0 & {}^nC_1\,{}^nC_2 + {}^nC_2\,{}^nC_3 + \dots + {}^nC_n\,{}^nC_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} {}^{2n}C_{n-1} & 0 \\ 0 & {}^{2n}C_{n-1} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$\therefore$ $a = {}^{2n}C_{n-1}$ $\quad \dfrac{a}{b} = 1, \quad a - b = 0$

$b = {}^{2n}C_{n-1}$

$$a+b = 2 \cdot {}^{2n}C_{n-1} = 2 \cdot \frac{2n!}{(n+1)!\,(n-1)!} = \frac{2n \cdot 2n!}{(n+1)!\; n!}$$

$$= \frac{2n}{n+1} \frac{2n!}{n!\,n!} = \frac{2n}{n+1}\, {}^{2n}C_n$$ **Ans**