2.7 Tangents and Derivatives

This section continues the discussion of secants and tangents begun in Section 2.1. We calculate limits of secant slopes to find tangents to curves.

What *Is* a Tangent to a Curve?

For circles, tangency is straightforward. A line L is tangent to a circle at a point P if L passes through P perpendicular to the radius at P (Figure 2.63). Such a line just *touches*

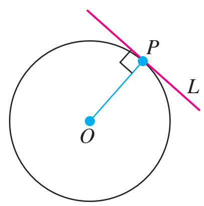
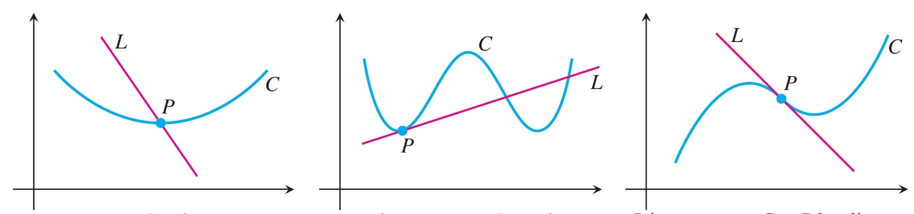


FIGURE 2.63 L is tangent to the circle at P if it passes through P perpendicular to radius OP.

the circle. But what does it mean to say that a line L is tangent to some other curve C at a point P? Generalizing from the geometry of the circle, we might say that it means one of the following:

1. L passes through P perpendicular to the line from P to the center of C.
2. L passes through only one point of C, namely P.
3. L passes through P and lies on one side of C only.

Although these statements are valid if C is a circle, none of them works consistently for more general curves. Most curves do not have centers, and a line we may want to call tangent may intersect C at other points or cross C at the point of tangency (Figure 2.64).



L meets C only at P but is not tangent to C.

L is tangent to C at P but meets C at several points.

L is tangent to C at P but lies on two sides of C, crossing C at P.

FIGURE 2.64 Exploding myths about tangent lines.

Historical Biography

Pierre de Fermat (1601–1665)

Biography

To define tangency for general curves, we need a *dynamic* approach that takes into account the behavior of the secants through P and nearby points Q as Q moves toward P along the curve (Figure 2.65). It goes like this:

1. We start with what we *can* calculate, namely the slope of the secant PQ.
2. Investigate the limit of the secant slope as Q approaches P along the curve.
3. If the limit exists, take it to be the slope of the curve at P and define the tangent to the curve at P to be the line through P with this slope.

This approach is what we were doing in the falling-rock and fruit fly examples in Section 2.1.

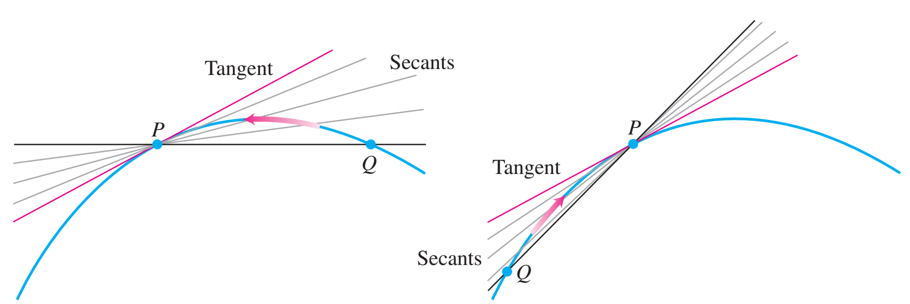


FIGURE 2.65 The dynamic approach to tangency. The tangent to the curve at P is the line through P whose slope is the limit of the secant slopes as $Q \to P$ from either side.

EXAMPLE 1 Tangent Line to a Parabola

Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$. Write an equation for the tangent to the parabola at this point.

Solution We begin with a secant line through $P(2, 4)$ and $Q(2 + h, (2 + h)^2)$ nearby. We then write an expression for the slope of the secant PQ and investigate what happens to the slope as Q approaches P along the curve:

$$\text{Secant slope} = \frac{\Delta y}{\Delta x} = \frac{(2 + h)^2 - 2^2}{h} = \frac{h^2 + 4h + 4 - 4}{h}$$

$$= \frac{h^2 + 4h}{h} = h + 4.$$

If $h > 0$, then Q lies above and to the right of P, as in Figure 2.66. If $h < 0$, then Q lies to the left of P (not shown). In either case, as Q approaches P along the curve, h approaches zero and the secant slope approaches 4:

$$\lim_{h \to 0} (h + 4) = 4.$$

We take 4 to be the parabola's slope at P.

The tangent to the parabola at P is the line through P with slope 4:

$$y = 4 + 4(x - 2) \qquad \text{Point-slope equation}$$

$$y = 4x - 4.$$

■

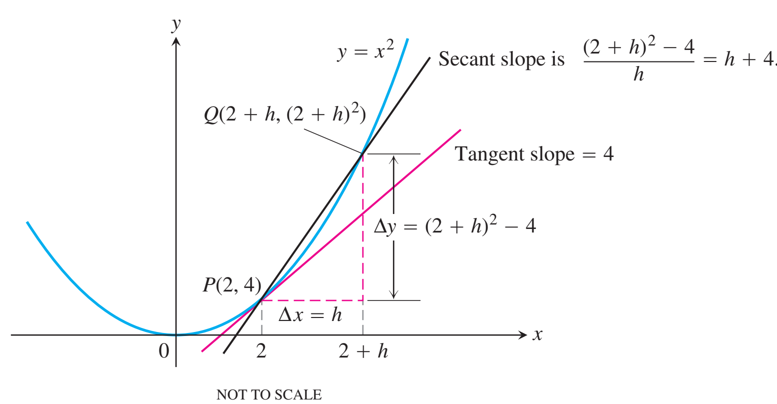


FIGURE 2.66 Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ (Example 1).

Finding a Tangent to the Graph of a Function

The problem of finding a tangent to a curve was the dominant mathematical problem of the early seventeenth century. In optics, the tangent determined the angle at which a ray of light entered a curved lens. In mechanics, the tangent determined the direction of a body's motion at every point along its path. In geometry, the tangents to two curves at a point of intersection determined the angles at which they intersected. To find a tangent to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$, we use the same dynamic procedure. We calculate the slope of the secant through P and a point $Q(x_0 + h, f(x_0 + h))$. We then investigate the limit of the slope as $h \to 0$ (Figure 2.67). If the limit exists, we call it the slope of the curve at P and define the tangent at P to be the line through P having this slope.

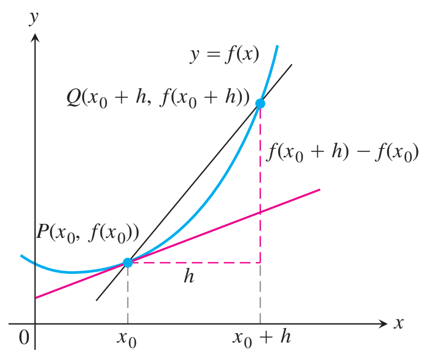


FIGURE 2.67 The slope of the tangent line at P is $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$.

DEFINITIONS Slope, Tangent Line

The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \quad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

Whenever we make a new definition, we try it on familiar objects to be sure it is consistent with results we expect in familiar cases. Example 2 shows that the new definition of slope agrees with the old definition from Section 1.2 when we apply it to nonvertical lines.

EXAMPLE 2 Testing the Definition

Show that the line $y = mx + b$ is its own tangent at any point $(x_0, mx_0 + b)$.

Solution We let $f(x) = mx + b$ and organize the work into three steps.

1. *Find $f(x_0)$ and $f(x_0 + h)$.*

$$f(x_0) = mx_0 + b$$
$$f(x_0 + h) = m(x_0 + h) + b = mx_0 + mh + b$$

2. *Find the slope $\lim_{h\to 0}(f(x_0 + h) - f(x_0))/h$.*

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} \frac{(mx_0 + mh + b) - (mx_0 + b)}{h}$$
$$= \lim_{h\to 0} \frac{mh}{h} = m$$

3. *Find the tangent line using the point-slope equation.* The tangent line at the point $(x_0, mx_0 + b)$ is

$$y = (mx_0 + b) + m(x - x_0)$$
$$y = mx_0 + b + mx - mx_0$$
$$y = mx + b.$$

■

Let's summarize the steps in Example 2.

Finding the Tangent to the Curve $y = f(x)$ at (x_0, y_0)

1. Calculate $f(x_0)$ and $f(x_0 + h)$.

2. Calculate the slope

$$m = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}.$$

3. If the limit exists, find the tangent line as

$$y = y_0 + m(x - x_0).$$

EXAMPLE 3 Slope and Tangent to $y = 1/x, x \neq 0$

(a) Find the slope of the curve $y = 1/x$ at $x = a \neq 0$.

(b) Where does the slope equal $-1/4$?

(c) What happens to the tangent to the curve at the point $(a, 1/a)$ as a changes?

Solution

(a) Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$\begin{aligned}\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\frac{a-(a+h)}{a(a+h)} \\ &= \lim_{h\to 0} \frac{-h}{ha(a+h)} \\ &= \lim_{h\to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.\end{aligned}$$

Notice how we had to keep writing "$\lim_{h\to 0}$" before each fraction until the stage where we could evaluate the limit by substituting $h = 0$. The number a may be positive or negative, but not 0.

(b) The slope of $y = 1/x$ at the point where $x = a$ is $-1/a^2$. It will be $-1/4$ provided that

$$-\frac{1}{a^2} = -\frac{1}{4}.$$

This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. The curve has slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$ (Figure 2.68).

(c) Notice that the slope $-1/a^2$ is always negative if $a \neq 0$. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep (Figure 2.69). We see this situation again as $a \to 0^-$. As a moves away from the origin in either direction, the slope approaches 0^- and the tangent levels off. ■

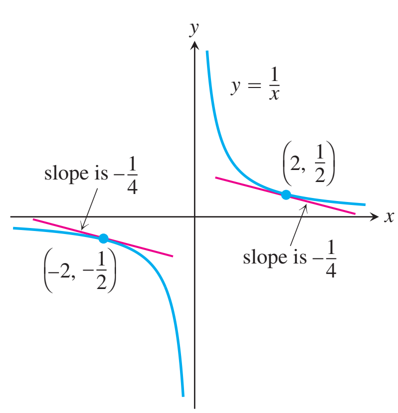


FIGURE 2.68 The two tangent lines to $y = 1/x$ having slope $-1/4$ (Example 3).

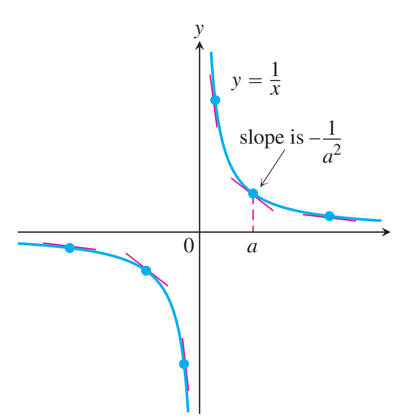


FIGURE 2.69 The tangent slopes, steep near the origin, become more gradual as the point of tangency moves away.

Rates of Change: Derivative at a Point

The expression

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

is called the **difference quotient of f at x_0 with increment h**. If the difference quotient has a limit as h approaches zero, that limit is called the **derivative of f at x_0**. If we interpret the difference quotient as a secant slope, the derivative gives the slope of the curve and tangent at the point where $x = x_0$. If we interpret the difference quotient as an average rate of change, as we did in Section 2.1, the derivative gives the function's rate of change with respect to x at the point $x = x_0$. The derivative is one of the two most important mathematical objects considered in calculus. We begin a thorough study of it in Chapter 3. The other important object is the integral, and we initiate its study in Chapter 5.

EXAMPLE 4 Instantaneous Speed (Continuation of Section 2.1, Examples 1 and 2)

In Examples 1 and 2 in Section 2.1, we studied the speed of a rock falling freely from rest near the surface of the earth. We knew that the rock fell $y = 16t^2$ feet during the first t sec, and we used a sequence of average rates over increasingly short intervals to estimate the rock's speed at the instant $t = 1$. Exactly what *was* the rock's speed at this time?

Solution We let $f(t) = 16t^2$. The average speed of the rock over the interval between $t = 1$ and $t = 1 + h$ seconds was

$$\frac{f(1 + h) - f(1)}{h} = \frac{16(1 + h)^2 - 16(1)^2}{h} = \frac{16(h^2 + 2h)}{h} = 16(h + 2).$$

The rock's speed at the instant $t = 1$ was

$$\lim_{h \to 0} 16(h + 2) = 16(0 + 2) = 32 \text{ ft/sec}.$$

Our original estimate of 32 ft/sec was right. ■

Summary

We have been discussing slopes of curves, lines tangent to a curve, the rate of change of a function, the limit of the difference quotient, and the derivative of a function at a point. All of these ideas refer to the same thing, summarized here:

1. The slope of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative of f at $x = x_0$
5. The limit of the difference quotient, $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$