Chapter 2 Additional and Advanced Exercises

T **1. Assigning a value to 0^0** The rules of exponents (see Appendix 9) tell us that $a^0 = 1$ if a is any number different from zero. They also tell us that $0^n = 0$ if n is any positive number.

If we tried to extend these rules to include the case 0^0, we would get conflicting results. The first rule would say $0^0 = 1$, whereas the second would say $0^0 = 0$.

We are not dealing with a question of right or wrong here. Neither rule applies as it stands, so there is no contradiction. We could, in fact, define 0^0 to have any value we wanted as long as we could persuade others to agree.

What value would you like 0^0 to have? Here is an example that might help you to decide. (See Exercise 2 below for another example.)

a. Calculate x^x for $x = 0.1, 0.01, 0.001$, and so on as far as your calculator can go. Record the values you get. What pattern do you see?

b. Graph the function $y = x^x$ for $0 < x \le 1$. Even though the function is not defined for $x \le 0$, the graph will approach the y-axis from the right. Toward what y-value does it seem to be headed? Zoom in to further support your idea.

T **2. A reason you might want 0^0 to be something other than 0 or 1** As the number x increases through positive values, the numbers $1/x$ and $1/(\ln x)$ both approach zero. What happens to the number

$$f(x) = \left(\frac{1}{x}\right)^{1/(\ln x)}$$

as x increases? Here are two ways to find out.

a. Evaluate f for $x = 10, 100, 1000$, and so on as far as your calculator can reasonably go. What pattern do you see?

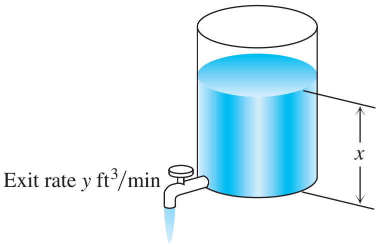
b. Graph f in a variety of graphing windows, including windows that contain the origin. What do you see? Trace the y-values along the graph. What do you find?

3. Lorentz contraction In relativity theory, the length of an object, say a rocket, appears to an observer to depend on the speed at which the object is traveling with respect to the observer. If the observer measures the rocket's length as L_0 at rest, then at speed v the length will appear to be

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}}.$$

This equation is the Lorentz contraction formula. Here, c is the speed of light in a vacuum, about 3×10^8 m/sec. What happens to L as v increases? Find $\lim_{v \to c^-} L$. Why was the left-hand limit needed?

4. Controlling the flow from a draining tank Torricelli's law says that if you drain a tank like the one in the figure shown, the rate y at which water runs out is a constant times the square root of the water's depth x. The constant depends on the size and shape of the exit valve.



Suppose that $y = \sqrt{x}/2$ for a certain tank. You are trying to maintain a fairly constant exit rate by adding water to the tank with a hose from time to time. How deep must you keep the water if you want to maintain the exit rate

a. within 0.2 ft^3/min of the rate $y_0 = 1\ \text{ft}^3/\text{min}$?

b. within 0.1 ft^3/min of the rate $y_0 = 1\ \text{ft}^3/\text{min}$?

5. Thermal expansion in precise equipment As you may know, most metals expand when heated and contract when cooled. The dimensions of a piece of laboratory equipment are sometimes so critical that the shop where the equipment is made must be held at the same temperature as the laboratory where the equipment is to be used. A typical aluminum bar that is 10 cm wide at 70°F will be

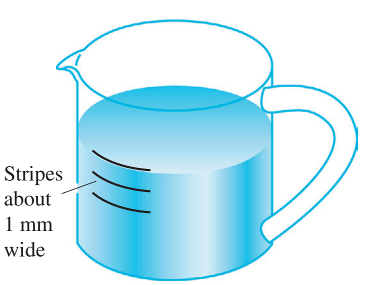
$$y = 10 + (t - 70) \times 10^{-4}$$

centimeters wide at a nearby temperature t. Suppose that you are using a bar like this in a gravity wave detector, where its width must stay within 0.0005 cm of the ideal 10 cm. How close to $t_0 = 70°\text{F}$ must you maintain the temperature to ensure that this tolerance is not exceeded?

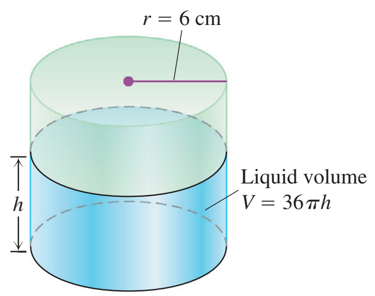
6. Stripes on a measuring cup The interior of a typical 1-L measuring cup is a right circular cylinder of radius 6 cm (see accompanying figure). The volume of water we put in the cup is therefore a function of the level h to which the cup is filled, the formula being

$$V = \pi 6^2 h = 36\pi h.$$

How closely must we measure h to measure out 1 L of water ($1000\ \text{cm}^3$) with an error of no more than 1% ($10\ \text{cm}^3$)?



(a)



(b)

A 1-L measuring cup (a), modeled as a right circular cylinder (b) of radius $r = 6$ cm

Precise Definition of Limit

In Exercises 7–10, use the formal definition of limit to prove that the function is continuous at x_0.

7. $f(x) = x^2 - 7, \quad x_0 = 1$ **8.** $g(x) = 1/(2x), \quad x_0 = 1/4$

9. $h(x) = \sqrt{2x - 3}, \quad x_0 = 2$ **10.** $F(x) = \sqrt{9 - x}, \quad x_0 = 5$

11. Uniqueness of limits Show that a function cannot have two different limits at the same point. That is, if $\lim_{x \to x_0} f(x) = L_1$ and $\lim_{x \to x_0} f(x) = L_2$, then $L_1 = L_2$.

12. Prove the limit Constant Multiple Rule:
$\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x)$ for any constant k.

13. One-sided limits If $\lim_{x \to 0^+} f(x) = A$ and $\lim_{x \to 0^-} f(x) = B$, find

a. $\lim_{x \to 0^+} f(x^3 - x)$ **b.** $\lim_{x \to 0^-} f(x^3 - x)$

c. $\lim_{x \to 0^+} f(x^2 - x^4)$ **d.** $\lim_{x \to 0^-} f(x^2 - x^4)$

14. Limits and continuity Which of the following statements are true, and which are false? If true, say why; if false, give a counterexample (that is, an example confirming the falsehood).

a. If $\lim_{x \to a} f(x)$ exists but $\lim_{x \to a} g(x)$ does not exist, then $\lim_{x \to a}(f(x) + g(x))$ does not exist.

b. If neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exists, then $\lim_{x \to a} (f(x) + g(x))$ does not exist.

c. If f is continuous at x, then so is $|f|$.

d. If $|f|$ is continuous at a, then so is f.

In Exercises 15 and 16, use the formal definition of limit to prove that the function has a continuous extension to the given value of x.

15. $f(x) = \dfrac{x^2 - 1}{x + 1}, \quad x = -1$ **16.** $g(x) = \dfrac{x^2 - 2x - 3}{2x - 6}, \quad x = 3$

17. A function continuous at only one point Let

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

a. Show that f is continuous at $x = 0$.

b. Use the fact that every nonempty open interval of real numbers contains both rational and irrational numbers to show that f is not continuous at any nonzero value of x.

18. The Dirichlet ruler function If x is a rational number, then x can be written in a unique way as a quotient of integers m/n where $n > 0$ and m and n have no common factors greater than 1. (We say that such a fraction is in *lowest terms*. For example, 6/4 written in lowest terms is 3/2.) Let $f(x)$ be defined for all x in the interval [0, 1] by

$$f(x) = \begin{cases} 1/n, & \text{if } x = m/n \text{ is a rational number in lowest terms} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

For instance, $f(0) = f(1) = 1$, $f(1/2) = 1/2$, $f(1/3) = f(2/3) = 1/3$, $f(1/4) = f(3/4) = 1/4$, and so on.

a. Show that f is discontinuous at every rational number in [0, 1].

b. Show that f is continuous at every irrational number in [0, 1]. (*Hint*: If ϵ is a given positive number, show that there are only finitely many rational numbers r in [0, 1] such that $f(r) \geq \epsilon$.)

c. Sketch the graph of f. Why do you think f is called the "ruler function"?

19. Antipodal points Is there any reason to believe that there is always a pair of antipodal (diametrically opposite) points on Earth's equator where the temperatures are the same? Explain.

20. If $\lim_{x \to c} (f(x) + g(x)) = 3$ and $\lim_{x \to c} (f(x) - g(x)) = -1$, find $\lim_{x \to c} f(x)g(x)$.

21. Roots of a quadratic equation that is almost linear The equation $ax^2 + 2x - 1 = 0$, where a is a constant, has two roots if $a > -1$ and $a \neq 0$, one positive and one negative:

$$r_+(a) = \frac{-1 + \sqrt{1 + a}}{a}, \qquad r_-(a) = \frac{-1 - \sqrt{1 + a}}{a}.$$

a. What happens to $r_+(a)$ as $a \to 0$? As $a \to -1^+$?

b. What happens to $r_-(a)$ as $a \to 0$? As $a \to -1^+$?

c. Support your conclusions by graphing $r_+(a)$ and $r_-(a)$ as functions of a. Describe what you see.

d. For added support, graph $f(x) = ax^2 + 2x - 1$ simultaneously for $a = 1, 0.5, 0.2, 0.1$, and 0.05.

22. Root of an equation Show that the equation $x + 2\cos x = 0$ has at least one solution.

23. Bounded functions A real-valued function f is **bounded from above** on a set D if there exists a number N such that $f(x) \leq N$ for all x in D. We call N, when it exists, an **upper bound** for f on D and say that f is bounded from above by N. In a similar manner, we say that f is **bounded from below** on D if there exists a number M such that $f(x) \geq M$ for all x in D. We call M, when it exists, a **lower bound** for f on D and say that f is bounded from below by M. We say that f is **bounded** on D if it is bounded from both above and below.

a. Show that f is bounded on D if and only if there exists a number B such that $|f(x)| \leq B$ for all x in D.

b. Suppose that f is bounded from above by N. Show that if $\lim_{x \to x_0} f(x) = L$, then $L \leq N$.

c. Suppose that f is bounded from below by M. Show that if $\lim_{x \to x_0} f(x) = L$, then $L \geq M$.

24. ***Max*** $\{a, b\}$ ***and min*** $\{a, b\}$

a. Show that the expression

$$\max\{a, b\} = \frac{a+b}{2} + \frac{|a-b|}{2}$$

equals a if $a \geq b$ and equals b if $b \geq a$. In other words, $\max\{a, b\}$ gives the larger of the two numbers a and b.

b. Find a similar expression for $\min\{a, b\}$, the smaller of a and b.

Generalized Limits Involving $\frac{\sin\theta}{\theta}$

The formula $\lim_{\theta\to 0} (\sin\theta)/\theta = 1$ can be generalized. If $\lim_{x\to c} f(x) = 0$ and $f(x)$ is never zero in an open interval containing the point $x = c$, except possibly c itself, then

$$\lim_{x\to c} \frac{\sin f(x)}{f(x)} = 1.$$

Here are several examples.

a. $\lim_{x\to 0} \frac{\sin x^2}{x^2} = 1.$

b. $\lim_{x\to 0} \frac{\sin x^2}{x} = \lim_{x\to 0} \frac{\sin x^2}{x^2} \lim_{x\to 0} \frac{x^2}{x} = 1 \cdot 0 = 0.$

c. $\lim_{x\to -1} \frac{\sin(x^2 - x - 2)}{x+1} = \lim_{x\to -1} \frac{\sin(x^2 - x - 2)}{(x^2 - x - 2)} \cdot$

$$\lim_{x\to -1} \frac{(x^2 - x - 2)}{x+1} = 1 \cdot \lim_{x\to -1} \frac{(x+1)(x-2)}{x+1} = -3.$$

d. $\lim_{x\to 1} \frac{\sin\left(1 - \sqrt{x}\right)}{x - 1} = \lim_{x\to 1} \frac{\sin\left(1 - \sqrt{x}\right)}{1 - \sqrt{x}} \frac{1 - \sqrt{x}}{x - 1} =$

$$1 \cdot \lim_{x\to 1} \frac{\left(1 - \sqrt{x}\right)\left(1 + \sqrt{x}\right)}{(x-1)\left(1 + \sqrt{x}\right)} = \lim_{x\to 1} \frac{1 - x}{(x-1)\left(1 + \sqrt{x}\right)} = -\frac{1}{2}.$$

Find the limits in Exercises 25–30.

25. $\lim_{x\to 0} \frac{\sin(1 - \cos x)}{x}$

26. $\lim_{x\to 0^+} \frac{\sin x}{\sin\sqrt{x}}$

27. $\lim_{x\to 0} \frac{\sin(\sin x)}{x}$

28. $\lim_{x\to 0} \frac{\sin(x^2 + x)}{x}$

29. $\lim_{x\to 2} \frac{\sin(x^2 - 4)}{x - 2}$

30. $\lim_{x\to 9} \frac{\sin\left(\sqrt{x} - 3\right)}{x - 9}$