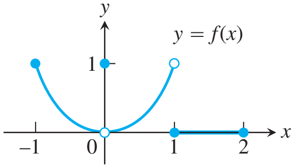
EXERCISES 2.4

Finding Limits Graphically

Exercises

1. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a. $\lim_{x \to -1^+} f(x) = 1$

b. $\lim_{x \to 0^-} f(x) = 0$

c. $\lim_{x \to 0^-} f(x) = 1$

d. $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

e. $\lim_{x \to 0} f(x)$ exists

f. $\lim_{x \to 0} f(x) = 0$

g. $\lim_{x \to 0} f(x) = 1$

h. $\lim_{x \to 1} f(x) = 1$

i. $\lim_{x \to 1} f(x) = 0$

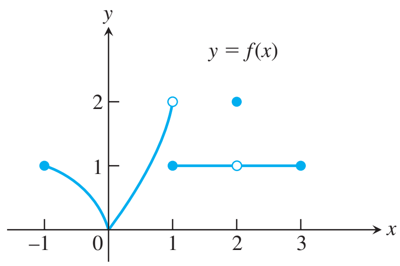
j. $\lim_{x \to 2^-} f(x) = 2$

k. $\lim_{x \to -1^-} f(x)$ does not exist.

l. $\lim_{x \to 2^+} f(x) = 0$

Exercises

2. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a. $\lim_{x\to -1^+} f(x) = 1$ **b.** $\lim_{x\to 2} f(x)$ does not exist.

c. $\lim_{x\to 2} f(x) = 2$ **d.** $\lim_{x\to 1^-} f(x) = 2$

e. $\lim_{x\to 1^+} f(x) = 1$ **f.** $\lim_{x\to 1} f(x)$ does not exist.

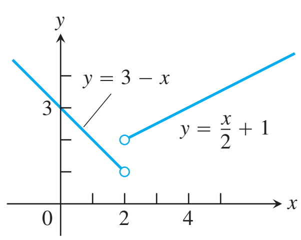
g. $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$

h. $\lim_{x\to c} f(x)$ exists at every c in the open interval $(-1, 1)$.

i. $\lim_{x\to c} f(x)$ exists at every c in the open interval $(1, 3)$.

j. $\lim_{x\to -1^-} f(x) = 0$ **k.** $\lim_{x\to 3^+} f(x)$ does not exist.

3. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{x}{2} + 1, & x > 2. \end{cases}$



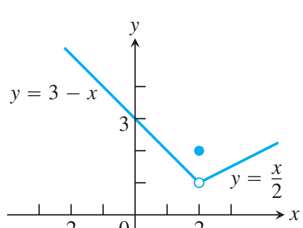
a. Find $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$.

b. Does $\lim_{x\to 2} f(x)$ exist? If so, what is it? If not, why not?

c. Find $\lim_{x\to 4^-} f(x)$ and $\lim_{x\to 4^+} f(x)$.

d. Does $\lim_{x\to 4} f(x)$ exist? If so, what is it? If not, why not?

4. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ \dfrac{x}{2}, & x > 2. \end{cases}$



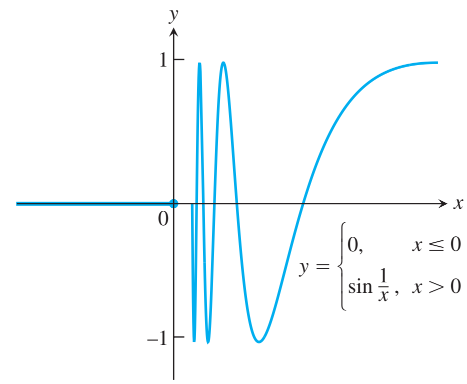
a. Find $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$, and $f(2)$.

b. Does $\lim_{x\to 2} f(x)$ exist? If so, what is it? If not, why not?

c. Find $\lim_{x\to -1^-} f(x)$ and $\lim_{x\to -1^+} f(x)$.

d. Does $\lim_{x\to -1} f(x)$ exist? If so, what is it? If not, why not?

5. Let $f(x) = \begin{cases} 0, & x \le 0 \\ \sin \dfrac{1}{x}, & x > 0. \end{cases}$

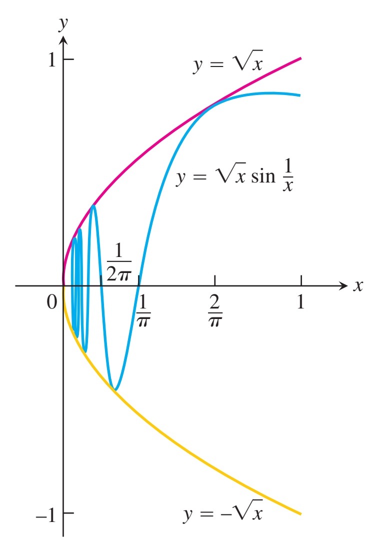


a. Does $\lim_{x\to 0^+} f(x)$ exist? If so, what is it? If not, why not?

b. Does $\lim_{x\to 0^-} f(x)$ exist? If so, what is it? If not, why not?

c. Does $\lim_{x\to 0} f(x)$ exist? If so, what is it? If not, why not?

6. Let $g(x) = \sqrt{x}\sin(1/x)$.



a. Does $\lim_{x\to 0^+} g(x)$ exist? If so, what is it? If not, why not?

b. Does $\lim_{x\to 0^-} g(x)$ exist? If so, what is it? If not, why not?

c. Does $\lim_{x\to 0} g(x)$ exist? If so, what is it? If not, why not?

7. **a.** Graph $f(x) = \begin{cases} x^3, & x \neq 1 \\ 0, & x = 1. \end{cases}$

b. Find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.

c. Does $\lim_{x\to 1} f(x)$ exist? If so, what is it? If not, why not?

8. **a.** Graph $f(x) = \begin{cases} 1 - x^2, & x \neq 1 \\ 2, & x = 1. \end{cases}$

b. Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$.

c. Does $\lim_{x\to 1} f(x)$ exist? If so, what is it? If not, why not?

Graph the functions in Exercises 9 and 10. Then answer these questions.

a. What are the domain and range of f?

b. At what points c, if any, does $\lim_{x\to c} f(x)$ exist?

c. At what points does only the left-hand limit exist?

d. At what points does only the right-hand limit exist?

9. $f(x) = \begin{cases} \sqrt{1 - x^2}, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \end{cases}$

10. $f(x) = \begin{cases} x, & -1 \le x < 0, \text{ or } 0 < x \le 1 \\ 1, & x = 0 \\ 0, & x < -1, \text{ or } x > 1 \end{cases}$

Finding One-Sided Limits Algebraically

Find the limits in Exercises 11–18.

11. $\lim_{x\to -0.5^-} \sqrt{\frac{x+2}{x+1}}$

12. $\lim_{x\to 1^+} \sqrt{\frac{x-1}{x+2}}$

13. $\lim_{x\to -2^+} \left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right)$

14. $\lim_{x\to 1^-} \left(\frac{1}{x+1}\right)\left(\frac{x+6}{x}\right)\left(\frac{3-x}{7}\right)$

15. $\lim_{h\to 0^+} \frac{\sqrt{h^2+4h+5} - \sqrt{5}}{h}$

16. $\lim_{h\to 0^-} \frac{\sqrt{6} - \sqrt{5h^2+11h+6}}{h}$

17. **a.** $\lim_{x\to -2^+} (x+3)\frac{|x+2|}{x+2}$ **b.** $\lim_{x\to -2^-} (x+3)\frac{|x+2|}{x+2}$

18. **a.** $\lim_{x\to 1^+} \frac{\sqrt{2x}\,(x-1)}{|x-1|}$ **b.** $\lim_{x\to 1^-} \frac{\sqrt{2x}\,(x-1)}{|x-1|}$

Use the graph of the greatest integer function $y = \lfloor x \rfloor$ (sometimes written $y = \operatorname{int} x$), Figure 1.31 in Section 1.3, to help you find the limits in Exercises 19 and 20.

19. **a.** $\lim_{\theta\to 3^+} \frac{\lfloor \theta \rfloor}{\theta}$ **b.** $\lim_{\theta\to 3^-} \frac{\lfloor \theta \rfloor}{\theta}$

20. **a.** $\lim_{t\to 4^+} (t - \lfloor t \rfloor)$ **b.** $\lim_{t\to 4^-} (t - \lfloor t \rfloor)$

Using $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

Find the limits in Exercises 21–36.

21. $\lim_{\theta\to 0} \frac{\sin\sqrt{2}\theta}{\sqrt{2}\theta}$

22. $\lim_{t\to 0} \frac{\sin kt}{t}$ (k constant)

23. $\lim_{y\to 0} \frac{\sin 3y}{4y}$

24. $\lim_{h\to 0^-} \frac{h}{\sin 3h}$

25. $\lim_{x\to 0} \frac{\tan 2x}{x}$

26. $\lim_{t\to 0} \frac{2t}{\tan t}$

27. $\lim_{x\to 0} \frac{x \csc 2x}{\cos 5x}$

28. $\lim_{x\to 0} 6x^2(\cot x)(\csc 2x)$

29. $\lim_{x\to 0} \frac{x + x\cos x}{\sin x \cos x}$

30. $\lim_{x\to 0} \frac{x^2 - x + \sin x}{2x}$

31. $\lim_{t\to 0} \frac{\sin(1-\cos t)}{1-\cos t}$

32. $\lim_{h\to 0} \frac{\sin(\sin h)}{\sin h}$

33. $\lim_{\theta\to 0} \frac{\sin\theta}{\sin 2\theta}$

34. $\lim_{x\to 0} \frac{\sin 5x}{\sin 4x}$

35. $\lim_{x\to 0} \frac{\tan 3x}{\sin 8x}$

36. $\lim_{y\to 0} \frac{\sin 3y \cot 5y}{y \cot 4y}$

Calculating Limits as $x \to \pm\infty$

In Exercises 37–42, find the limit of each function **(a)** as $x \to \infty$ and **(b)** as $x \to -\infty$. (You may wish to visualize your answer with a graphing calculator or computer.)

37. $f(x) = \frac{2}{x} - 3$

38. $f(x) = \pi - \frac{2}{x^2}$

39. $g(x) = \frac{1}{2 + (1/x)}$

40. $g(x) = \frac{1}{8 - (5/x^2)}$

41. $h(x) = \frac{-5 + (7/x)}{3 - (1/x^2)}$

42. $h(x) = \frac{3 - (2/x)}{4 + \left(\sqrt{2}/x^2\right)}$

Find the limits in Exercises 43–46.

43. $\lim_{x\to\infty} \frac{\sin 2x}{x}$

44. $\lim_{\theta\to -\infty} \frac{\cos\theta}{3\theta}$

45. $\lim_{t\to -\infty} \frac{2 - t + \sin t}{t + \cos t}$

46. $\lim_{r\to\infty} \frac{r + \sin r}{2r + 7 - 5\sin r}$

Limits of Rational Functions

In Exercises 47–56, find the limit of each rational function **(a)** as $x \to \infty$ and **(b)** as $x \to -\infty$.

47. $f(x) = \frac{2x+3}{5x+7}$

48. $f(x) = \frac{2x^3+7}{x^3 - x^2 + x + 7}$

49. $f(x) = \frac{x+1}{x^2+3}$

50. $f(x) = \frac{3x+7}{x^2-2}$

51. $h(x) = \frac{7x^3}{x^3 - 3x^2 + 6x}$

52. $g(x) = \frac{1}{x^3 - 4x + 1}$

53. $g(x) = \dfrac{10x^5 + x^4 + 31}{x^6}$

54. $h(x) = \dfrac{9x^4 + x}{2x^4 + 5x^2 - x + 6}$

55. $h(x) = \dfrac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x}$

56. $h(x) = \dfrac{-x^4}{x^4 - 7x^3 + 7x^2 + 9}$

Limits with Noninteger or Negative Powers

The process by which we determine limits of rational functions applies equally well to ratios containing noninteger or negative powers of x: divide numerator and denominator by the highest power of x in the denominator and proceed from there. Find the limits in Exercises 57–62.

57. $\lim_{x\to\infty} \dfrac{2\sqrt{x} + x^{-1}}{3x - 7}$

58. $\lim_{x\to\infty} \dfrac{2 + \sqrt{x}}{2 - \sqrt{x}}$

59. $\lim_{x\to-\infty} \dfrac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}}$

60. $\lim_{x\to\infty} \dfrac{x^{-1} + x^{-4}}{x^{-2} - x^{-3}}$

61. $\lim_{x\to\infty} \dfrac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}}$

62. $\lim_{x\to-\infty} \dfrac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4}$

Theory and Examples

63. Once you know $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ at an interior point of the domain of f, do you then know $\lim_{x\to a} f(x)$? Give reasons for your answer.

64. If you know that $\lim_{x\to c} f(x)$ exists, can you find its value by calculating $\lim_{x\to c^+} f(x)$? Give reasons for your answer.

65. Suppose that f is an odd function of x. Does knowing that $\lim_{x\to 0^+} f(x) = 3$ tell you anything about $\lim_{x\to 0^-} f(x)$? Give reasons for your answer.

66. Suppose that f is an even function of x. Does knowing that $\lim_{x\to 2^-} f(x) = 7$ tell you anything about either $\lim_{x\to -2^-} f(x)$ or $\lim_{x\to -2^+} f(x)$? Give reasons for your answer.

67. Suppose that $f(x)$ and $g(x)$ are polynomials in x and that $\lim_{x\to\infty} (f(x)/g(x)) = 2$. Can you conclude anything about $\lim_{x\to-\infty} (f(x)/g(x))$? Give reasons for your answer.

68. Suppose that $f(x)$ and $g(x)$ are polynomials in x. Can the graph of $f(x)/g(x)$ have an asymptote if $g(x)$ is never zero? Give reasons for your answer.

69. How many horizontal asymptotes can the graph of a given rational function have? Give reasons for your answer.

70. Find $\lim_{x\to\infty} \left(\sqrt{x^2 + x} - \sqrt{x^2 - x}\right)$.

Use the formal definitions of limits as $x \to \pm\infty$ to establish the limits in Exercises 71 and 72.

71. If f has the constant value $f(x) = k$, then $\lim_{x\to\infty} f(x) = k$.

72. If f has the constant value $f(x) = k$, then $\lim_{x\to-\infty} f(x) = k$.

Formal Definitions of One-Sided Limits

73. Given $\epsilon > 0$, find an interval $I = (5, 5 + \delta)$, $\delta > 0$, such that if x lies in I, then $\sqrt{x - 5} < \epsilon$. What limit is being verified and what is its value?

74. Given $\epsilon > 0$, find an interval $I = (4 - \delta, 4)$, $\delta > 0$, such that if x lies in I, then $\sqrt{4 - x} < \epsilon$. What limit is being verified and what is its value?

Use the definitions of right-hand and left-hand limits to prove the limit statements in Exercises 75 and 76.

75. $\lim_{x\to 0^-} \dfrac{x}{|x|} = -1$

76. $\lim_{x\to 2^+} \dfrac{x - 2}{|x - 2|} = 1$

77. **Greatest integer function** Find **(a)** $\lim_{x\to 400^+} \lfloor x \rfloor$ and **(b)** $\lim_{x\to 400^-} \lfloor x \rfloor$; then use limit definitions to verify your findings. **(c)** Based on your conclusions in parts (a) and (b), can anything be said about $\lim_{x\to 400} \lfloor x \rfloor$? Give reasons for your answers.

78. **One-sided limits** Let $f(x) = \begin{cases} x^2 \sin(1/x), & x < 0 \\ \sqrt{x}, & x > 0. \end{cases}$

Find **(a)** $\lim_{x\to 0^+} f(x)$ and **(b)** $\lim_{x\to 0^-} f(x)$; then use limit definitions to verify your findings. **(c)** Based on your conclusions in parts (a) and (b), can anything be said about $\lim_{x\to 0} f(x)$? Give reasons for your answer.

Grapher Explorations—"Seeing" Limits at Infinity

Sometimes a change of variable can change an unfamiliar expression into one whose limit we know how to find. For example,

$$\begin{aligned} \lim_{x\to\infty} \sin\frac{1}{x} &= \lim_{\theta\to 0^+} \sin\theta \qquad \text{Substitute } \theta = 1/x \\ &= 0. \end{aligned}$$

This suggests a creative way to "see" limits at infinity. Describe the procedure and use it to picture and determine limits in Exercises 79–84.

79. $\lim_{x\to\pm\infty} x \sin\dfrac{1}{x}$

80. $\lim_{x\to-\infty} \dfrac{\cos(1/x)}{1 + (1/x)}$

81. $\lim_{x\to\pm\infty} \dfrac{3x + 4}{2x - 5}$

82. $\lim_{x\to\infty} \left(\dfrac{1}{x}\right)^{1/x}$

83. $\lim_{x\to\pm\infty} \left(3 + \dfrac{2}{x}\right)\left(\cos\dfrac{1}{x}\right)$

84. $\lim_{x\to\infty} \left(\dfrac{3}{x^2} - \cos\dfrac{1}{x}\right)\left(1 + \sin\dfrac{1}{x}\right)$