Trigonometric Equation

1. Solve for θ the equation $\sin^2\theta = \sin^2\alpha$

Soln. $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha, \; n \in I$

2. Solve the equation $\tan^2 x + \cot^2 x = 2$

Soln: $\tan^2 x + \dfrac{1}{\tan^2 x} = 2 \Rightarrow (\tan^2 x)^2 - 2\tan^2 x + 1 = 0$

$(\tan^2 x - 1) = 0 \Rightarrow \tan^2 x = 1 \Rightarrow \tan^2 x = \tan^2 \frac{\pi}{4}$

$\Rightarrow x = n\pi \pm \frac{\pi}{4}$

3. Solve $2 + 7\tan^2\theta = 3 \cdot 25 \sec^2\theta$

Soln. $2 + 7\tan^2\theta = \frac{13}{4}(1 + \tan^2\theta)$

$\Rightarrow 7\tan^2\theta - \frac{13}{4}\tan^2\theta = \frac{13}{4} - 2$

$\Rightarrow \frac{15}{4}\tan^2\theta = \frac{5}{4} \Rightarrow \tan^2\theta = 3$

$\Rightarrow \tan^2\theta = \tan^2\frac{\pi}{3}$ ..

$\Rightarrow \theta = n\pi \pm \frac{\pi}{3}, \; n \in I$

4. Solve : $3\tan\theta + \cot\theta = 5\operatorname{cosec}\theta$

Soln :- $\dfrac{3\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} = \dfrac{5}{\sin\theta} \Rightarrow 3\sin^2\theta + \cos^2\theta = 5\cos\theta$

$\Rightarrow 2\sin^2\theta + 1 = 5\cos\theta$

$\Rightarrow 2(1 - \cos^2\theta) + 1 = 5\cos\theta \Rightarrow 2\cos^2\theta + 5\cos\theta - 3 = 0$

$\Rightarrow 2\cos^2\theta + 6\cos\theta - \cos\theta - 3 = 0$

$\Rightarrow 2\cos\theta(\cos\theta + 3) - (\cos\theta + 3) = 0$

$\Rightarrow (\cos\theta + 3)(2\cos\theta - 1) = 0$ But $(\cos\theta + 3) \neq 0$

$\therefore 2\cos\theta - 1 = 0 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \cos\theta = \cos\frac{\pi}{3}$

$\therefore \theta = 2n\pi \pm \pi/3, \; n \in I$

5. Find all values of θ satisfying the equation $\sin\theta + \sin 5\theta = \sin 3\theta$, such that $0 \le \theta \le \pi$.

Soln. $\sin\theta + \sin 5\theta = \sin 3\theta$

$\Rightarrow 2\sin 3\theta \cos 2\theta = \sin 3\theta \Rightarrow 2\sin 3\theta \cos 2\theta - \sin 3\theta = 0$

$\Rightarrow \sin 3\theta (2\cos 2\theta - 1) = 0$

$\Rightarrow \sin 3\theta = 0$ or, $2\cos 2\theta - 1 = 0$

$3\theta = n\pi$, $\quad \cos 2\theta = \frac{1}{2}$

$\theta = \frac{n\pi}{3}$, $\quad \cos 2\theta = \cos\frac{\pi}{3}$.

$2\theta = 2n\pi \pm \frac{\pi}{3}$.

$\theta = n\pi \pm \frac{\pi}{6}$, $n \in I$

Solution set $\left\{ \frac{n\pi}{3}, n\pi \pm \frac{\pi}{6} \right\}$, $n \in I$

6. Find all values of θ satisfying the equation

(i) $\cos 6\theta + \cos 4\theta + \cos 2\theta + 1 = 0$

soln: $2\cos 5\theta \cos\theta + 2\cos^2\theta = 0$

$\Rightarrow 2\cos\theta(\cos 5\theta + \cos\theta) = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2}$ — (i)

or, $\cos 5\theta + \cos\theta = 0$

$2\cos 3\theta \cos 2\theta = 0 \Rightarrow \cos 2\theta = 0 \Rightarrow 2\theta = (2n+1)\frac{\pi}{2}$

$\Rightarrow \theta = (2n+1)\frac{\pi}{4}$ — (ii)

and when $\cos 3\theta = 0 \Rightarrow 3\theta = (2n+1)\frac{\pi}{2} \Rightarrow \theta = (2n+1)\frac{\pi}{6}$ — (iii)

Solution set $\left\{ (2n+1)\frac{\pi}{2}, (2n+1)\frac{\pi}{4}, (2n+1)\frac{\pi}{6}, n \in I \right\}$

(ii) $\cos\theta + \cos 2\theta + \cos 3\theta = 0$

Soln. $\cos 3\theta + \cos\theta + \cos 2\theta = 0$

$\Rightarrow 2\cos 2\theta \cos\theta + \cos 2\theta = 0 \Rightarrow \cos 2\theta(2\cos\theta + 1) = 0$

$\Rightarrow \cos 2\theta = 0$ or $2\cos\theta + 1 = 0$

$\Rightarrow 2\theta = (2n+1)\frac{\pi}{2}$ or, $\cos\theta = -\frac{1}{2} \Rightarrow \cos\theta = \cos\frac{2\pi}{3}$.

$\Rightarrow \theta = 2n\pi \pm \frac{2\pi}{3}$.

Solution Set $\left\{ (2n+1)\frac{\pi}{2},\ 2n\pi \pm \frac{2\pi}{3},\ n \in I \right\}$

(7) Find the values of x between 0 and 2π, satisfying the equation $\cos 3x + \cos 2x = \sin \frac{3x}{2} + \sin \frac{x}{2}$

Soln

$$2\cos\frac{5x}{2}\cos\frac{x}{2} = 2\sin x \cos\frac{x}{2}$$

$$\Rightarrow \cos\frac{x}{2}\left[\cos\frac{5x}{2} - \sin x\right] = 0$$

$$\Rightarrow \cos\frac{x}{2} = 0 \quad \text{or} \quad \cos\frac{5x}{2} - \sin x = 0$$

$$\Rightarrow \frac{x}{2} = (2n+1)\frac{\pi}{2}, \quad \cos\frac{5x}{2} = \sin x \Rightarrow \cos\frac{5x}{2} - \cos\left(\frac{\pi}{2} - x\right) = 0$$

$$\Rightarrow x = (2n+1)\pi, \quad -2\sin\left(\frac{\frac{5x}{2} + \frac{\pi}{2} - x}{2}\right)\sin\left(\frac{\frac{5x}{2} - \frac{\pi}{2} + x}{2}\right) = 0$$

$$\Rightarrow \sin\left(\frac{\pi}{4} + \frac{3x}{4}\right)\sin\left(\frac{7x}{4} - \frac{\pi}{4}\right) = 0$$

$$\Rightarrow \sin\left(\frac{3x}{4} + \frac{\pi}{4}\right) = 0 \Rightarrow \frac{3x + \pi}{4} = n\pi$$

$$\Rightarrow 3x = \frac{(4n-1)\pi}{3}$$

$$\Rightarrow \sin\left(\frac{7x - \pi}{4}\right) = 0 \Rightarrow \frac{7x - \pi}{4} = n\pi$$

$$\Rightarrow 7x - \pi = 4n\pi$$

$$\Rightarrow x = \frac{(4n+1)}{7}\pi$$

$\therefore$ Solution Set $\left\{ (2n+1)\pi,\ \frac{(4n-1)}{3}\pi,\ \frac{(4n+1)}{7}\pi \right\}$

(8) Solve the equation

$$\sin 2x + \cos 2x + \sin x + \cos x + 1 = 0$$

Soln:

$$\sin 2x + \sin x + (\cos 2x + 1) + \cos x = 0$$

$$\Rightarrow 2\sin x \cos x + \sin x + 2\cos^2 x + \cos x = 0$$

$$\Rightarrow \sin x(2\cos x + 1) + \cos x(2\cos x + 1) = 0$$

$$\Rightarrow (2\cos x + 1)(\sin x + \cos x) = 0$$

$\Rightarrow$ either $2\cos x + 1 = 0$ or $\sin x + \cos x = 0$

$\Rightarrow$ $\cos x = -\frac{1}{2}$ or $\sin x = -\cos x$.

$\Rightarrow$ $\cos x = \cos\frac{2\pi}{3}$ or, $\tan x = -1$

$\Rightarrow$ $x = 2n\pi \pm \frac{2\pi}{3}$ or, $\tan x = \tan\frac{3\pi}{4}$

$x = n\pi + \frac{3\pi}{4}$

$\therefore$ Solution Set $\left\{2n\pi \pm \frac{2\pi}{3},\ n\pi + \frac{3\pi}{4}\right\}, n \in I$

9. Solve the equations

(i) $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$

Soln. $\sin 3x + \sin x + \sin 2x = \cos 3x + \cos x + \cos 2x$

$\Rightarrow 2\sin 2x \cos x + \sin 2x = 2\cos 2x \cos x + \cos 2x$

$\Rightarrow \sin 2x(2\cos x + 1) - \cos 2x(2\cos x + 1) = 0$

$\Rightarrow (2\cos x + 1)(\sin 2x - \cos 2x) = 0$

$\Rightarrow$ either $2\cos x + 1 = 0$ or, $\sin 2x - \cos 2x = 0$

$\Rightarrow$ $\cos x = -\frac{1}{2}$ or, $\sin 2x = \cos 2x$

$\Rightarrow$ $\cos x = \cos\frac{2\pi}{3}$ or, $\tan 2x = 1$

$\Rightarrow$ $x = 2n\pi \pm \frac{2\pi}{3}$ or $2x = n\pi + \frac{\pi}{4}$

$x = \frac{n\pi}{2} + \frac{\pi}{8}$

$\therefore$ Solution Set $\left\{2n\pi \pm \frac{2\pi}{3},\ \frac{n\pi}{2} + \frac{\pi}{8}\right\}$ $n \in I$

(ii) $\cos 6x + \cos 4x = \sin 3x + \sin x$

Soln. $2\cos 5x \cos x = 2\sin 2x \cos x$

$$2\cos x(\cos 5x - \sin 2x) = 0$$

$$\Rightarrow \cos x = 0 \quad \text{or} \quad \cos 5x - \sin 2x = 0$$

$$\Rightarrow x = (2n+1)\frac{\pi}{2}, \quad \text{or}, \quad \cos 5x - \cos\left(\frac{\pi}{2} - 2x\right) = 0$$

$$\Rightarrow -2\sin\left(\frac{5x + \frac{\pi}{2} - 2x}{2}\right)\sin\left(\frac{5x - \frac{\pi}{2} + 2x}{2}\right) = 0$$

$$\Rightarrow \sin\left(\frac{3x}{2} + \frac{\pi}{4}\right)\sin\left(\frac{7x}{2} - \frac{\pi}{4}\right) = 0$$

when $\sin\left(\frac{3x}{2} + \frac{\pi}{4}\right) = 0 \Rightarrow \frac{3x}{2} + \frac{\pi}{4} = n\pi \Rightarrow x = \frac{2}{3}\left(n\pi - \frac{\pi}{4}\right)$

$$\Rightarrow x = (4n-1)\frac{\pi}{6}, \quad n \in I$$

when $\sin\left(\frac{7x}{2} - \frac{\pi}{4}\right) = 0 \Rightarrow \frac{7x}{2} - \frac{\pi}{4} = n\pi, \Rightarrow \frac{7x}{\cancel{2}} = \frac{(4n+1)\pi}{\cancel{4}\,2}$

$$\Rightarrow x = (4n+1)\frac{\pi}{14}, \quad n \in I$$

$\therefore$ Solution set $\left\{(2n+1)\frac{\pi}{2}, (4n-1)\frac{\pi}{6}, (4n+1)\frac{\pi}{14} \cdot n \in I\right\}$

10. Find the general solution of $\sec 4\theta - \sec 2\theta = 2$

Soln- $\frac{1}{\cos 4\theta} - \frac{1}{\cos 2\theta} = 2 \Rightarrow \frac{\cos 2\theta - \cos 4\theta}{\cos 4\theta \cos 2\theta} = 2$

$$\cos 2\theta - \cos 4\theta = 2\cos 4\theta \cos 2\theta$$

$$\cos 2\theta = 2\cos 4\theta \cos 2\theta + \cos 4\theta.$$

$$= 2\cos 2\theta(2\cos^2 2\theta - 1) + 2\cos^2 2\theta - 1$$

$$\cos 2\theta = 4\cos^3 2\theta - 2\cos 2\theta + 2\cos^2 2\theta - 1$$

$$\Rightarrow 4\cos^3 2\theta + 2\cos^2 2\theta - 3\cos 2\theta - 1 = 0$$

$$\Rightarrow (\cos 2\theta + 1)(4\cos^2 2\theta - 2\cos 2\theta - 1) = 0$$

when $\cos 2\theta + 1 = 0 \Rightarrow \cos 2\theta = -1$

$$\Rightarrow \cos 2\theta = \cos \pi$$

$$\Rightarrow 2\theta = 2n\pi \pm \pi$$

$$\Rightarrow \theta = (2n \pm 1)\frac{\pi}{2} \quad \text{--- (I)}$$

When $4\cos^2 2\theta - 2\cos 2\theta - 1 = 0$

$$\therefore \cos 2\theta = \frac{2 \pm \sqrt{4+16}}{2 \times 4} = \frac{2 \pm 2\sqrt{5}}{2 \times 4} = \frac{1 \pm \sqrt{5}}{4}$$

Again

$$\cos 2\theta = \frac{\sqrt{5}+1}{4}$$

$$\cos 2\theta = \cos\frac{\pi}{5}$$

$$2\theta = 2n\pi \pm \frac{\pi}{5}$$

$$\theta = n\pi \pm \frac{\pi}{10} \quad \text{--- (II)}$$

$$\cos 2\theta = \frac{1-\sqrt{5}}{4} = \frac{-(\sqrt{5}-1)}{4}$$

$$\cos 2\theta = -\cos 72^\circ$$

$$\cos 2\theta = \cos(180 - 72^\circ)$$

$$= \cos 108^\circ$$

$$\cos 2\theta = \cos\frac{3\pi}{5}$$

$$\therefore 2\theta = 2n\pi \pm \frac{3\pi}{5}$$

$$\therefore \theta = n\pi \pm \frac{3\pi}{10} \quad \text{--- (III)}$$

$\therefore$ Solution set $\left\{(2n \pm 1)\frac{\pi}{2},\ n\pi \pm \frac{\pi}{10},\ n\pi \pm \frac{3\pi}{10},\ n \in I\right\}$

11. Find all the angle θ between $-\pi$ and π that satisfy the equation $5\cos 2\theta + 2\cos^2\frac{\theta}{2} + 1 = 0$

Soln:

$$5\cos 2\theta + 2\cos^2\frac{\theta}{2} + 1 = 0$$

$$\Rightarrow 5(2\cos^2\theta - 1) + (1 + \cos\theta) + 1 = 0$$

$$\Rightarrow 10\cos^2\theta + \cos\theta - 3 = 0$$

$$\Rightarrow 10\cos^2\theta + 6\cos\theta - 5\cos\theta - 3 = 0$$

$$\Rightarrow 2\cos\theta(5\cos\theta + 3) - (5\cos\theta + 3) = 0$$

$$\Rightarrow (2\cos\theta - 1)(5\cos\theta + 3) = 0$$

when $2\cos\theta - 1 = 0 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \cos\theta = \cos\frac{\pi}{3}$.

$\Rightarrow \theta = 2n\pi \pm \frac{\pi}{3}$ —— (i)

when $5\cos\theta + 3 = 0 \Rightarrow \cos\theta = \frac{-3}{5} \Rightarrow \theta = 2n\pi \pm \cos^{-1}\left(\frac{-3}{5}\right)$ —— (ii)

$\therefore$ Solution Set $\left\{2n\pi \pm \frac{\pi}{3},\ 2n\pi \pm \cos^{-1}\left(-\frac{3}{5}\right), n \in I\right\}$

But values lies between $-\pi$ to π are

$-\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \cos^{-1}\left(-\frac{3}{5}\right)$

12. Find all the values of θ between 0 and $\frac{\pi}{2}$, satisfying the equation $\sin 7\theta + \sin 4\theta + \sin\theta = 0$

Soln:

$\sin 7\theta + \sin\theta + \sin 4\theta = 0$

$\Rightarrow 2\sin 4\theta \cos 3\theta + \sin 4\theta = 0$

$\Rightarrow \sin 4\theta\,(2\cos 3\theta + 1) = 0$

when $\sin 4\theta = 0 \Rightarrow 4\theta = \pi \Rightarrow \theta = \frac{\pi}{4}$ —— (1)

when $2\cos 3\theta + 1 = 0 \Rightarrow \cos 3\theta = -\frac{1}{2} \Rightarrow \cos 3\theta = \cos\frac{2\pi}{3}$

$\Rightarrow 3\theta = \frac{2\pi}{3}, \frac{4\pi}{3} \quad \therefore \theta = \frac{2\pi}{9}, \frac{4\pi}{9}$

$\therefore$ Solution Set $\left\{\frac{\pi}{4}, \frac{2\pi}{9}, \frac{4\pi}{9}\right\}$

13. If $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$ Prove That

$\cos\left(\theta - \frac{\pi}{4}\right) = \pm\frac{1}{2\sqrt{2}}$

Soln:

$\tan(\pi\cos\theta) = \tan\left(\pm\frac{\pi}{2} - \pi\sin\theta\right)$

$\Rightarrow \quad \pi\cos\theta = \pm\frac{\pi}{2} - \pi\sin\theta$

$\Rightarrow \quad \cos\theta + \sin\theta = \pm\frac{1}{2}$

$\Rightarrow \quad \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta = \pm\frac{1}{2\sqrt{2}}$

$\Rightarrow \quad \cos\theta\cos\frac{\pi}{4} + \sin\theta\sin\frac{\pi}{4} = \pm\frac{1}{2\sqrt{2}}$

$\Rightarrow \quad \cos\left(\theta - \frac{\pi}{4}\right) = \pm\frac{1}{2\sqrt{2}}$ proved.

14. If α and β be two distinct roots of the equation

$a\tan\theta + b\sec\theta = c$

prove that $\tan(\alpha+\beta) = \dfrac{2ac}{a^2-c^2}$

Soln. $a\tan\theta + b\sec\theta = c$

$a\sin\theta + b = c\cos\theta \Rightarrow a\sin\theta - c\cos\theta = b$ —(1)

$\because$ α and β are roots of the equation

$\therefore$ $a\sin\alpha - c\cos\alpha = b$

$a\sin\beta - c\cos\beta = b$

$\Rightarrow \quad a(\sin\alpha - \sin\beta) - c(\cos\alpha - \cos\beta) = 0$

$\Rightarrow \quad a\, 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} + c\, 2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} = 0$

$\Rightarrow \quad 2\sin\left(\frac{\alpha-\beta}{2}\right)\left[a\cos\left(\frac{\alpha+\beta}{2}\right) + c\sin\left(\frac{\alpha+\beta}{2}\right)\right] = 0$

$\because$ $\sin\left(\frac{\alpha-\beta}{2}\right) \neq 0, \quad \because \alpha \neq \beta.$

$\therefore$ $a\cos\left(\frac{\alpha+\beta}{2}\right) + c\sin\left(\frac{\alpha+\beta}{2}\right) = 0$

$\tan\left(\frac{\alpha+\beta}{2}\right) = -\frac{a}{c}$ —— (1)

$\boxed{\because \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}}$

$$\tan(\alpha+\beta) = \frac{2\tan\left(\frac{\alpha+\beta}{2}\right)}{1-\tan^2\left(\frac{\alpha+\beta}{2}\right)} = \frac{2\left(-\frac{a}{c}\right)}{1-\frac{a^2}{c^2}} = \frac{2ac}{a^2-c^2}$$

15. If $\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$ Prove That

(i) $\cos\left(\theta \pm \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$

(ii) $\sin 2\theta = \pm\frac{3}{4}$

Soln

$$\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$$

$$\Rightarrow \sin(\pi\cos\theta) = \sin\left(\frac{\pi}{2} \pm \pi\sin\theta\right)$$

$$\Rightarrow \pi\cos\theta = \frac{\pi}{2} \pm \pi\sin\theta$$

$$\Rightarrow \cos\theta \pm \sin\theta = \frac{1}{2} \Rightarrow \frac{1}{\sqrt{2}}\cos\theta \pm \frac{1}{\sqrt{2}}\sin\theta = \frac{1}{2\sqrt{2}}$$

$$\Rightarrow \cos\theta\cos\frac{\pi}{4} \pm \sin\theta\sin\frac{\pi}{4} = \frac{1}{2\sqrt{2}}$$

$$\Rightarrow \cos\left(\theta \pm \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$$

(ii) We have $\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$

We deduce as $\sin\theta \pm \cos\theta = \frac{1}{2}$

Squaring both the sides we get

$$\sin^2\theta + \cos^2\theta \pm 2\sin\theta\cos\theta = \frac{1}{4}$$

$$\Rightarrow 1 \pm \sin 2\theta = \frac{1}{4} \Rightarrow \mp\sin 2\theta = 1 - \frac{1}{4}$$

$$\Rightarrow \mp\sin 2\theta = \frac{3}{4} \Rightarrow \sin 2\theta = \pm\frac{3}{4}$$

Proved.

16. Find the maximum and minimum value of $7\cos\theta + 24\sin\theta$

Soln: We know $a\sin\theta + b\cos\theta$ lies in

$$-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$$

$$\therefore -\sqrt{7^2+24^2} \le 7\cos\theta + 24\sin\theta \le \sqrt{7^2+24^2}$$

$$-25 \le 7\cos\theta + 24\sin\theta \le 25$$

$\therefore$ Max. value $= 25$, min value $= -25$

17. Prove that $5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right) + 3$ lies between -4 and 10

Soln. $5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right) + 3$

$$= 5\cos\theta + 3\left(\cos\theta\cos\frac{\pi}{3} + \sin\theta\sin\frac{\pi}{3}\right) + 3.$$

$$= 5\cos\theta + 3\left(\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right) + 3.$$

$$= 5\cos\theta + \frac{3}{2}\cos\theta + \frac{3\sqrt{3}}{2}\sin\theta + 3$$

$$= \frac{13}{2}\cos\theta + \frac{3\sqrt{3}}{2}\sin\theta + 3.$$

But,

$$-\sqrt{\left(\frac{13}{2}\right)^2+\left(\frac{3\sqrt{3}}{2}\right)^2} \le \frac{13}{2}\cos\theta + \frac{3\sqrt{3}}{2}\sin\theta \le \sqrt{\left(\frac{13}{2}\right)^2+\left(\frac{3\sqrt{3}}{2}\right)^2}$$

$$3-\sqrt{\frac{169+27}{4}} \le \frac{13}{2}\cos\theta + \frac{3\sqrt{3}}{2}\sin\theta + 3 \le \sqrt{\frac{169+27}{4}} + 3$$

$$-4 \le \frac{13}{2}\cos\theta + \frac{3\sqrt{3}}{2}\sin\theta + 3 \le 10$$ Proved.

18. Find a and b such that the inequality holds good for all θ,

$$a \le 3\cos\theta + 5\sin\left(\theta - \frac{\pi}{6}\right) \le b.$$

Soln.

$$a \le 3\cos\theta + 5\left(\sin\theta \cdot \frac{\sqrt{3}}{2} - \frac{1}{2}\cos\theta\right) \le b.$$

$$a \le 3\cos\theta + \frac{5\sqrt{3}}{2}\sin\theta - \frac{5}{2}\cos\theta \le b$$

$$a \le \frac{1}{2}\cos\theta + \frac{5\sqrt{3}}{2}\sin\theta \le b \quad \text{--- (1)}$$

$$\left[\because -\sqrt{a^2+b^2} \le a\cos\theta + b\sin\theta \le \sqrt{a^2+b^2}\right]$$

$$\therefore -\sqrt{\frac{1}{4}+\frac{75}{4}} \le \frac{1}{2}\cos\theta + \frac{5\sqrt{3}}{2}\sin\theta \le \sqrt{\frac{1}{4}+\frac{75}{4}}$$

$$-\sqrt{19} \le \frac{1}{2}\cos\theta + \frac{5\sqrt{3}}{2}\sin\theta \le \sqrt{19}$$

comparing with (1), we have $a = -\sqrt{19}$
$b = \sqrt{19}$

19. Show that for all real θ, the expression $a\sin^2\theta + b\sin\theta\cos\theta + c\cos^2\theta$ lies between $\frac{1}{2}\left\{(a+c) - \sqrt{b^2+(a-c)^2}\right\}$ and $\frac{1}{2}\left\{(a+c) + \sqrt{b^2+(a-c)^2}\right\}$

Soln. we know $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\therefore a\sin^2\theta + b\sin\theta\cos\theta + c\cos^2\theta$$

$$= a\left(\frac{1-\cos 2\theta}{2}\right) + \frac{b}{2}\,2\sin\theta\cos\theta + c\left(\frac{1+\cos 2\theta}{2}\right)$$

$$= \frac{a}{2} + \frac{c}{2} + \left(\frac{c}{2} - \frac{a}{2}\right)\cos 2\theta + \frac{b}{2}\sin 2\theta$$

$$= \frac{a+c}{2} + \frac{c-a}{2}\cos 2\theta + \frac{b}{2}\sin 2\theta$$

$\because$

$$-\sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{c-a}{2}\right)^2} \le \frac{b}{2}\sin 2\theta + \frac{c-a}{2}\cos 2\theta \le \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{c-a}{2}\right)^2}$$

$$\therefore \frac{a+c}{2} - \frac{1}{2}\sqrt{b^2+(c-a)^2} \le \frac{c+a}{2} + \frac{b}{2}\sin 2\theta + \left(\frac{c-a}{2}\right)\cos 2\theta \le \frac{a+c}{2} + \frac{1}{2}\sqrt{b^2+(c-a)^2}$$

Proved

20. Show that for varying θ and fixed α, the expression $\frac{\tan(\theta+\alpha)}{\tan(\theta-\alpha)}$ can not lie between $\tan^2\left(\frac{\pi}{4}-\alpha\right)$ and $\tan^2\left(\frac{\pi}{4}+\alpha\right)$.

Soln. Let $y = \frac{\tan(\theta+\alpha)}{\tan(\theta-\alpha)}$

$$\Rightarrow \frac{y+1}{y-1} = \frac{\tan(\theta+\alpha) + \tan(\theta-\alpha)}{\tan(\theta+\alpha) - \tan(\theta-\alpha)}$$

$$\Rightarrow \frac{y+1}{y-1} = \frac{\frac{\sin(\theta+\alpha)}{\cos(\theta+\alpha)} + \frac{\sin(\theta-\alpha)}{\cos(\theta-\alpha)}}{\frac{\sin(\theta+\alpha)}{\cos(\theta+\alpha)} - \frac{\sin(\theta-\alpha)}{\cos(\theta-\alpha)}}$$

$$\Rightarrow \frac{y+1}{y-1} = \frac{\sin(\theta+\alpha)\cos(\theta-\alpha) + \cos(\theta+\alpha)\sin(\theta-\alpha)}{\sin(\theta+\alpha)\cos(\theta-\alpha) - \cos(\theta+\alpha)\sin(\theta-\alpha)}$$

$$\Rightarrow \frac{y+1}{y-1} = \frac{\sin(\theta+\cancel{\alpha}+\theta-\cancel{\alpha})}{\sin(\theta+\alpha-\theta+\alpha)} = \frac{\sin 2\theta}{\sin 2\alpha}$$

$$\Rightarrow \frac{y+1}{y-1}\sin 2\alpha = \sin 2\theta$$

$$\Rightarrow \frac{y+1}{y-1}\sin 2\alpha = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$\Rightarrow \left[\frac{(y+1)}{y-1}(\sin 2\alpha)\right]\tan^2\theta - 2\tan\theta + \left(\frac{y+1}{y-1}\right)\sin 2\alpha = 0$$

for real value of θ, $D \geq 0 \Rightarrow b^2 \geq 4ac$

$$4 \geq 4\left[\left(\frac{y+1}{y-1}\right)\sin 2\alpha\right]^2$$

$$\Rightarrow \left[\left(\frac{y+1}{y-1}\right)\sin 2\alpha\right]^2 - 1 \leq 0$$

$$-1 \leq \frac{y+1}{y-1}\sin 2\alpha \leq 1$$

when $\frac{y+1}{y-1}\sin 2\alpha \leq 1$

$$\Rightarrow y\sin 2\alpha + \sin 2\alpha - y + 1 \leq 0$$

$$1+\sin 2\alpha \leq y(1-\sin 2\alpha)$$

$$\therefore y \geq \frac{1+\sin 2\alpha}{1-\sin 2\alpha} = \left(\frac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha}\right)^2$$

$$= \left(\frac{1+\tan\alpha}{1-\tan\alpha}\right)^2$$

$$= \tan^2\left(\frac{\pi}{4}+\alpha\right)$$

Similarly, when $-1 \leq \frac{y+1}{y-1}\sin 2\alpha \Rightarrow y \leq \tan^2\left(\frac{\pi}{4}-\alpha\right)$

$\therefore$ y does not lies between $\tan^2\left(\frac{\pi}{4}-\alpha\right)$ and $\tan^2\left(\frac{\pi}{4}+\alpha\right)$