Definite Integration

CONCEPT NOTES

Definite Integration

As explained in the chapter titled "**Integration Basics**", the fundamental theorem of calculus tells us that to evaluate the area under a curve $y = f(x)$ from $x = a$ to $x = b$, we first evaluate the anti-derivative $g(x)$ of $f(x)$

$$g(x) = \int f(x)\,dx$$

and then evaluate $g(b) - g(a)$. That is, area under the curve $f(x)$ from $x = a$ to $x = b$ is

$$\int_a^b f(x)\,dx = g(b) - g(a)$$

Readers who have even the slightest doubt regarding the discussion above are advised to refer to the chapter on "**Integration Basics**" before reading on.

Definite integration is not all about just evaluating the anti-derivative and substituting the upper and lower limits. Working through this chapter, you will realise that a lot of techniques exist which help us in evaluating the definite integral without resorting to the (many times tedious) process of first determining the anti-derivative. We will develop all these techniques one by one from scratch, starting with some extremely basic properties in Section - 1

Section - 1 | BASIC PROPERTIES

(1) Suppose that $f(x) < 0$ on some interval $[a, b]$. Then, the area under the curve $y = f(x)$ from $x = a$ to $x = b$ will be negative in sign, i.e

$$\int_a^b f(x)\,dx < 0$$

This is obvious once you consider how the definite integral was arrived at in the first place; as a limit of the sum of the n rectangles $(n \to \infty)$. Thus, if $f(x) < 0$ in some interval then the area of the rectangles in that interval will also be negative.

This property means that for example, if $f(x)$ has the following form

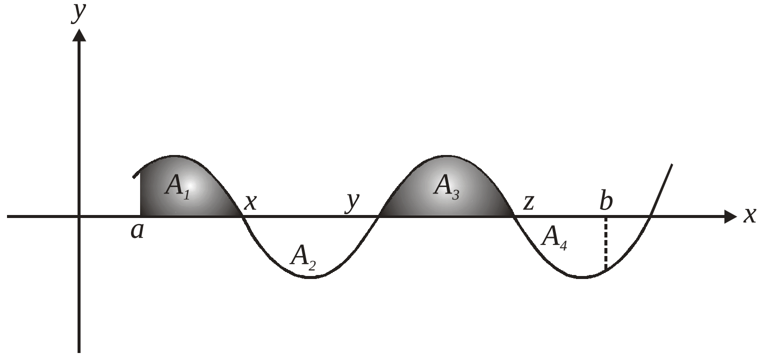


Fig - 1

then $\int_a^b f(x)\,dx$ will equal $A_1 - A_2 + A_3 - A_4$ and not $A_1 + A_2 + A_3 + A_4$.

If we need to evaluate $A_1 + A_2 + A_3 + A_4$ (the magnitude of the bounded area), we will have to calculate

$$\int_a^x f(x)dx + \left|\int_x^y f(x)dx\right| + \int_y^z f(x)dx + \left|\int_z^b f(x)dx\right|$$

From this, it should also be obvious that $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$

(2) The area under the curve $y = f(x)$ from $x = a$ to $x = b$ is equal in magnitude but opposite in sign to the area under the same curve from $x = b$ to $x = a$, i.e

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

This property is obvious if you consider the Newton-Leibnitz formula. If $g(x)$ is the anti-derivative of $x(f)$, then $\int_a^b f(x)dx$ is $g(b) - g(a)$ while $\int_b^a f(x)dx$ is $g(a) - g(b)$.

(3) The area under the curve $y = f(x)$ from $x = a$ to $x = b$ can be written as the sum of the area under the curve from $x = a$ to $x = c$ and from $x = c$ to $x = b$, that is

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Let us consider an example of this. Let $c \in (a, b)$

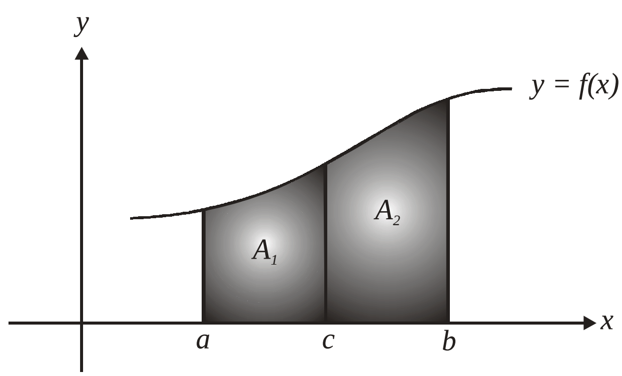


Fig - 2

It is clear that the area under the curve from $x = a$ to $x = b$, A is $A_1 + A_2$.

Note that c need not lie between a and b for this relation to hold true. Suppose that $c > b$.

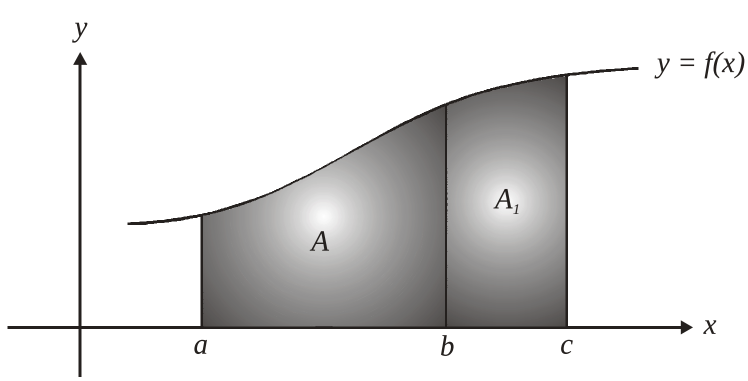


Fig - 3

Observe that $A = \int_a^b f(x)dx = (A + A_1) - A_1$

$$= \int_a^c f(x)dx - \int_b^c f(x)dx$$

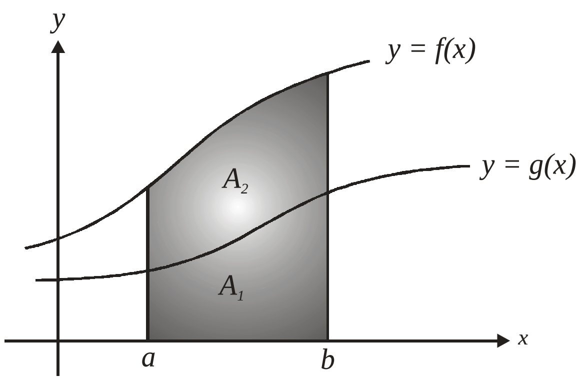
$$= \int_a^c f(x)dx + \int_c^b f(x)dx$$

Analytically, this relation can be proved easily using the Newton Leibnitz's formula.

(4) Let $f(x) > g(x)$ on the interval [a, b]. Then,

$$\int_a^b f(x) > \int_a^b g(x)dx.$$

This is because the curve of $f(x)$ lies above the curve of $g(x)$, or equivalently, the curve of $f(x) - g(x)$ lies above the x-axis for [a, b]



This is an example where f(x) > g(x) > 0.

$$\int_a^b f(x)dx = A_1 + A_2$$

while $\int_a^b g(x)dx = A_1$

Fig - 4

Similarly, if $f(x) < g(x)$ on the interval [a, b], then

$$\int_a^b f(x)dx < \int_a^b g(x)dx$$

(5) For the interval [a, b], suppose $m < f(x) < M$. That is, m is a lower-bound for $f(x)$ while M is an upper bound.

Then,

$$m(b-a) < \int_a^b f(x)dx < M(b-a)$$

This is obvious once we consider the figure below:

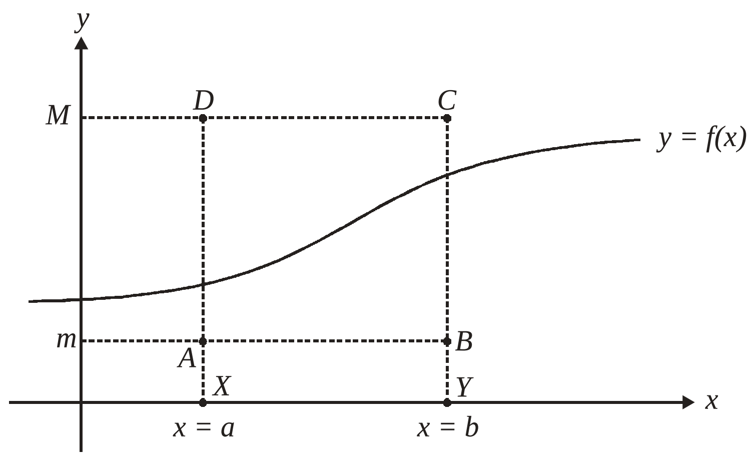


Fig - 5

Observe that $\text{area}\left(\text{rect } AXYB\right) < \int_a^b f\left(x\right)dx < \text{area}\left(\text{rect } DXYC\right)$

(6) Let us consider the integral of $f_1\left(x\right)+f_2\left(x\right)$ from $x=a$ to $x=b$. To evaluate the area under $f_1\left(x\right)+f_2\left(x\right)$, we can separately evaluate the area under $f_1(x)$ and the area under $f_2(x)$ and add the two areas (algebraically). Thus:

$$\int_a^b \left(f_1\left(x\right)+f_2\left(x\right)\right)dx = \int_a^b f_1\left(x\right)dx + \int_a^b f_2\left(x\right)dx$$

Now consider the integral of $kf(x)$ from $x = a$ to $x = b$. To evaluate the area under $kf(x)$, we can first evaluate the area under $f(x)$ and then multiply it by k, that is:

$$\int_a^b kf\left(x\right)dx = k\int_a^b f\left(x\right)dx$$

(7) Consider an odd function $f(x)$, *i.e*, $f\left(x\right)=-f\left(-x\right)$. This means that the graph of $f(x)$ is symmetric about the origin.

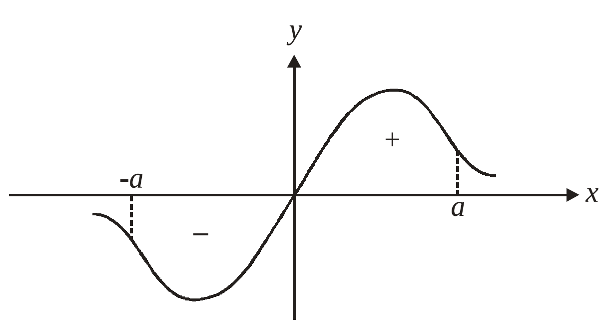


Fig - 6

From the figure, it should be obvious that $\int_{-a}^{a} f\left(x\right)dx = 0,$ because the area on the left side and that on the right algebraically add to 0.

Similarly, if $f(x)$ was even, *i.e,* $f(x) = f(-x)$

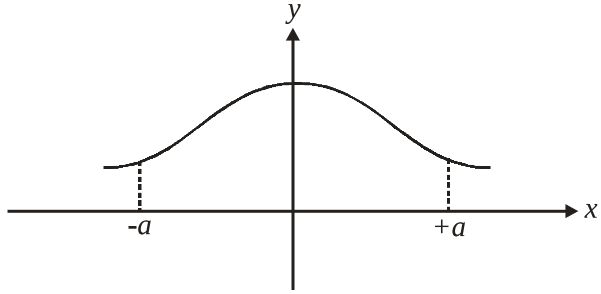
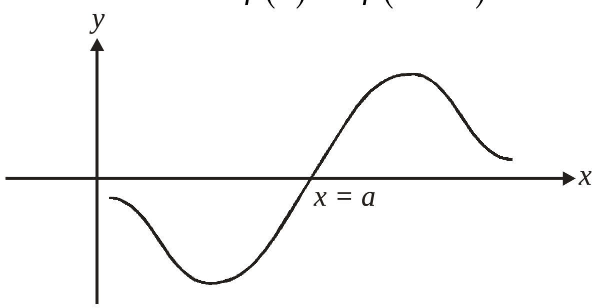


Fig - 7

$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ because the graph is symmetrical about the y-axis.

If you recall the discussion in the unit on functions, a function can also be even or odd about any arbitrary point $x = a$. Let us suppose that $f(x)$ is odd about $x = a$, i.e

$$f(x) = -f(2a - x)$$



The points x and 2a-x lie equidistant from x = a at either sides of it.

Fig - 8

Suppose for example, that we need to calculate $\int_{0}^{2a} f(x)\,dx.$ It is obvious that this will be 0, since we are considering equal variation on either side of $x = a$, i.e. the area from $x = 0$ to $x = a$ and the area from $x = a$ to $x = 2a$ will add algebraically to 0.

Similarly, if $f(x)$ is even about $x = a$, i.e.

$$f(x) = f(2a - x)$$

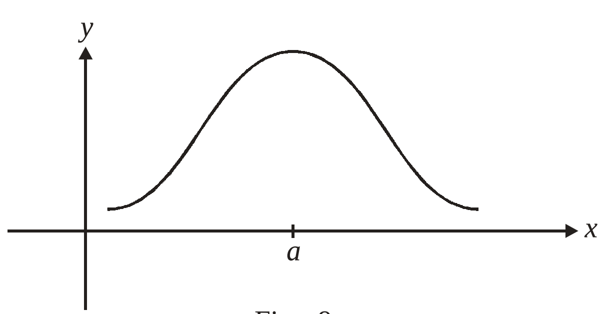


Fig - 9

then we have, for example

$$\int_{0}^{2a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

From this discussion, you will get a general idea as to how to approach such issues regarding even/odd functions.

(8) Let us consider a function $f(x)$ on $[a, b]$

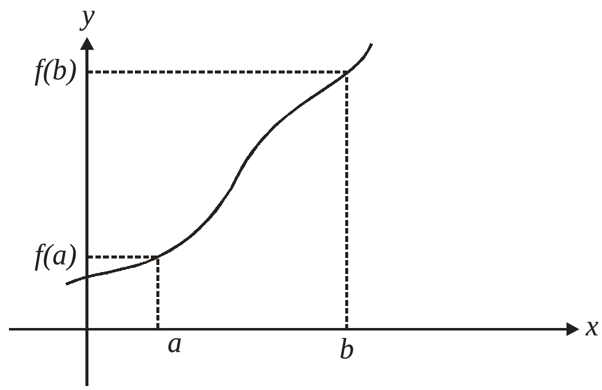


Fig - 10

We want to somehow define the "average" value that $f(x)$ takes on the interval $[a, b]$. What would be an appropriate way to define such an average?

Let f_{av} be the average value that we are seeking. Let it be such that it is obtained at some $x = c \in [a, b]$

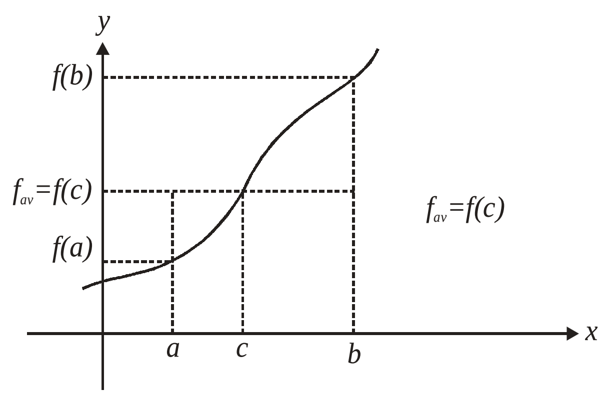


Fig - 11

We can measure f_{av} by saying that the area under $f(x)$ from $x = a$ to $x = b$ should equal the area under the average value from $x = a$ to $x = b$. This seems to be the only logical way to define the average (and this is how it is actually defined!). Thus

$$f_{av}(b-a) = \int_a^b f(x)\,dx$$

$$\Rightarrow f_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

This value is attained for at least one $c \in (a, b)$ (under the constraint that f is continuous, of course).

Example – 1

Find the area under the curve $y=\dfrac{1}{x^2+1}$ on its entire domain, i.e., evaluate $\displaystyle\int_{-\infty}^{+\infty}\frac{1}{x^2+1}\,dx$

Solution: The graph for $y=\dfrac{1}{x^2+1}$ is sketched below:

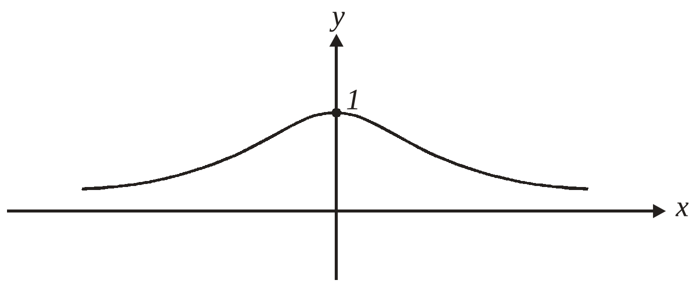


Fig - 12

Although the graph extends to infinity on both sides, the area under the curve will still be finite, as we'll now see:

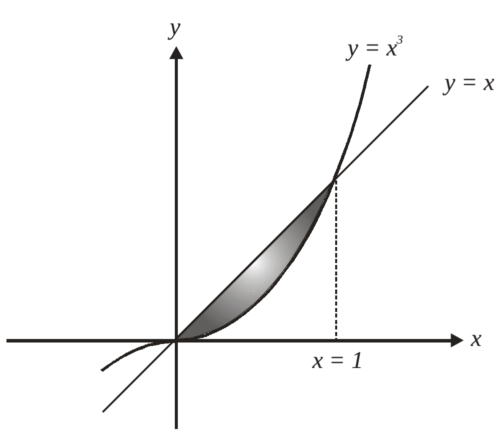
$$\int_{-\infty}^{+\infty}\frac{1}{x^2+1}\,dx = \left.\tan^{-1}x\right|_{-\infty}^{+\infty}$$
$$=\tan^{-1}(+\infty)-\tan^{-1}(-\infty)$$

What do we make of $\tan^{-1}(+\infty)$? We can take it to mean $\lim\limits_{k\to\infty}\tan^{-1}(k)$ which is $\dfrac{\pi}{2}$. Similarly, $\tan^{-1}(-\infty)$ would equal $\dfrac{-\pi}{2}$. Thus, the required area is π. ◀

Example – 2

Evaluate the area bounded between $y=x$ and $y=x^3$ from $x=0$ to $x=1$.

Solution: The given curves are sketched in the region of interest below.



The two curves intersect when $x = x^3$, i.e at $x = 1$ (on the positive side). Since $x^3 < x$ for $(0, 1)$, the graph of x^3 lies below that of x

Fig - 13

The required area is

$$A=\int_0^1\left(x-x^3\right)dx$$

$$=\left(\frac{x^2}{2}-\frac{x^4}{2}\right)\Bigg|_0^1$$

$$=\left(\frac{1}{2}-\frac{1}{4}\right)-(0-0)$$

$$=\frac{1}{4}$$

◀

Example – 3

Find the mean value of $f(x)=\cos^2 x$ on $\left[0,\frac{\pi}{2}\right]$

Solution: Let f_{av} be the required mean value.

$$\Rightarrow f_{av}\left(\frac{\pi}{2}-0\right)=\int_0^{\pi/2}\cos^2 x\,dx$$

$$=\frac{1}{2}\int_0^{\pi/2}(1+\cos 2x)\,dx$$

$$=\frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)\Bigg|_0^{\pi/2}$$

$$=\frac{\pi}{4}$$

$$\Rightarrow f_{av}=\frac{1}{2}$$

◀

Example – 4

Evaluate $\displaystyle\int_{-\infty}^{+\infty}\frac{1}{\left(x^2+a^2\right)\left(x^2+b^2\right)}dx$

Solution: $\dfrac{1}{\left(x^2+a^2\right)\left(x^2+b^2\right)}$ can be rewritten as $\dfrac{1}{\left(b^2-a^2\right)}\left\{\dfrac{1}{x^2+a^2}-\dfrac{1}{x^2+b^2}\right\}$

Therefore, the given integral becomes

$$\frac{1}{\left(b^2-a^2\right)}\left\{\int_{-\infty}^{+\infty}\frac{1}{\left(x^2+a^2\right)}dx-\int_{-\infty}^{+\infty}\frac{1}{\left(x^2+b^2\right)}dx\right\}$$

$$=\frac{1}{\left(b^2-a^2\right)}\left\{\frac{1}{a}\tan^{-1}\frac{x}{a}\Bigg|_{-\infty}^{+\infty}-\frac{1}{b}\tan^{-1}\frac{x}{b}\Bigg|_{-\infty}^{+\infty}\right\}$$

$$= \frac{1}{\left(b^2 - a^2\right)} \left\{ \frac{\pi}{a} - \frac{\pi}{b} \right\}$$

$$= \frac{\pi}{ab(a+b)}$$ ◀

Sometimes, to modify an integral, an appropriate substitution has to be used; the same way we did in the unit on **Indefinite Integration**. For example, integrals containing the expression $(x^2 + a^2)$ can be simplified (or modified) using the substitution $x = a\tan\theta$.

For evaluating a definite integral too, we can use the appropriate substitution, provided we change the limits of integration accordingly also. This will become clear in subsequent examples.

Example – 5

If $I_n = \int_0^{x/4} \tan^n x \, dx,$ prove that

(a) $I_n + I_{n-2} = \frac{1}{n-1}$ **(b)** $\frac{1}{n+1} < 2I_n < \frac{1}{n-1}$

Solution: (a)

$$I_n + I_{n-2} = \int_0^{\pi/4} \tan^n x \, dx + \int_0^{\pi/4} \tan^{n-2} x \, dx$$

$$= \int_0^{\pi/4} \left(\tan^n x + \tan^{n-2} x\right) dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x \left(\tan^2 x + 1\right) dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x \sec^2 x \; dx$$

The substitution $\tan x = t$ can now be used to simplify this integral. However, we must change the limits of integration according to this substitution:

$$\tan x = t \quad \Rightarrow \quad \sec^2 x \, dx = dt$$

$$\text{If} \quad x = 0 \;\Rightarrow\; t = 0$$

$$\text{If} \quad x = \frac{\pi}{4} \;\Rightarrow\; t = 1$$

Thus, the modified integral (in terms of the new variable t) is:

$$I_n + I_{n-2} = \int_0^1 t^{n-2} \, dt$$

$$= \left.\frac{t^{n-1}}{n-1}\right|_0^1$$

$$= \frac{1}{n-1}$$ ◀

(b) In the integral that we are considering, the limits of integration are 0 to $\frac{\pi}{4}$, *i.e*, $x \in \left[0, \frac{\pi}{4}\right]$

In this interval, $\tan x < 1$. Thus,

$$\tan^{n+2} x < \tan^n x < \tan^{n-2} x \quad \forall \quad x \in \left(0, \frac{\pi}{4}\right)$$

From property (4), we can therefore say that:

$$\int_0^{\pi/4} \tan^{n+2} x\,dx < \int_0^{\pi/4} \tan^n x < \int_0^{\pi/4} \tan^{n-2} x$$

or

$$I_{n+2} < I_n < I_{n-2}$$

$$\Rightarrow I_n + I_{n+2} < 2I_n < I_n + I_{n-2} \qquad \text{... (1)}$$

Using the result of part (a) for the first and third terms in (1), we get our desired result:

$$\frac{1}{n+1} < 2I_n < \frac{1}{n-1}$$

◀

Example – 6

For $x > 0$, let $f(x) = \int_1^x \frac{\ln t}{1+t}\,dt$. Find the function $f(x) + f\left(\frac{1}{x}\right)$ and show that $f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2}$.

Solution: Observe carefully the form of the function $f(x)$: It is in the form of an integral (of another function), with the lower limit being fixed and the upper limit being the variable x. As x varies, $f(x)$ will correspondingly vary.

One approach that you might contemplate to solve this question is evaluate the anti-derivative $g(t)$ of $\frac{\ln t}{1+t}$ and then evaluate $g(x) - g(1)$ which will become $f(x)$. However, this will become unnecessarily cumbersome (Try it!). We can, instead, proceed as follows:

$$f(x) + f\left(\frac{1}{x}\right) = \int_1^x \frac{\ln t}{1+t}\,dt + \int_1^{1/x} \frac{\ln t}{1+t}\,dt$$

$$= \quad I_1 \quad + \quad I_2$$

Notice that the limits of integration of I_1 and I_2 are different. If they were the same, we could have added I_1 and I_2 easily. So we try to make them the same: in I_2, if we let $t = \frac{1}{y}$, and t varies from 1 to $\frac{1}{x}$, y will vary from 1 to x. This substitution will therefore make the limits of integration of I_2 the same as those of I_1:

$$t = \frac{1}{y}$$

$$dt = -\frac{1}{y^2}\,dy$$

$$t = 1 \;\Rightarrow\; y = 1$$

$$t = \frac{1}{x} \Rightarrow y = x$$

$$I_2 = \int_1^{1/x} \frac{\ln t}{1+t}\,dt$$

$$= -\int_1^{x} \frac{\ln(1/y)}{1+(1/y)}\,\frac{1}{y^2}\,dy$$

$$= \int_1^{x} \frac{\ln y}{y(1+y)}\,dy$$

I_1 and I_2 can now be easily added:

$$= I_1 + I_2 = \int_1^{x} \left\{ \frac{\ln t}{1+t} + \frac{\ln t}{t(1+t)} \right\} dt$$

$$= \int_1^{x} \frac{(1+t)\ln t}{t(1+t)}\,dt$$

$$= \int_1^{x} \frac{\ln t}{t}\,dt$$

{We used t instead of y in I_2. This doesn't make a difference; y is the variable of integration; it can be replaced with any other variable as long as the limits of integration are the same.}

The final expression shows how simplified $I_1 + I_2$ has become. We let $\ln t = z \Rightarrow \frac{1}{t}\,dt = dz$ and the limits of integration become 0 to $\ln x$.

$$I_1 + I_2 = \int_0^{\ln x} z\,dz$$

$$= \frac{1}{2}(\ln x)^2$$

Thus,

$$f(e) + f\left(\frac{1}{e}\right) = \left(\frac{\ln e}{2}\right)^2$$

$$= \frac{1}{2}$$

◀

Example – 7

Evaluate $\int_0^{\pi/2} \left(\sqrt{\sin x} + \sqrt{\cos x}\right)^{-4} dx$

Solution: The given integral can be modified into an (easily) integrable form by expressing it in a form involving tan *x* and sec *x*.

$$I = \int_0^{\pi/2} \frac{1}{\left(\sqrt{\sin x} + \sqrt{\cos x}\right)^4} dx$$

$$= \int_0^{\pi/2} \frac{1}{\cos^2 x\left(1 + \sqrt{\tan x}\right)^4} dx$$

$$= \int_0^{\pi/2} \frac{\sec^2 x}{\left(1 + \sqrt{\tan x}\right)^4} dx$$

The substitution $\tan x = y^2$ can now be used.

$$\Rightarrow \sec^2 x\, dx = 2y dy$$

$$x = 0 \Rightarrow y = 0$$

$$x = \pi/2 \Rightarrow y = \infty$$

$$I = 2\int_0^{\infty} \frac{y\, dy}{(1+y)^4}$$

$$= 2\int_0^{\infty} \frac{(1+y) - 1}{(1+y)^4} dy$$

$$= 2\int_0^{\infty} \left\{ \frac{1}{(1+y)^3} - \frac{1}{(1+y)^4} \right\} dy$$

$$= 2\left\{ \left. \frac{(1+y)^{-2}}{-2} \right|_0^{\infty} - \left. \frac{(1+y)^{-3}}{-3} \right|_0^{\infty} \right\}$$

$$= 2\left(\frac{1}{2} - \frac{1}{3} \right)$$

$$= \frac{1}{3}$$

◀

Example – 8

Determine a positive integer $n \le 5$ such that

$$\int_0^1 e^x (x-1)^n \, dx = 16 - 6e$$

Solution: Since n will not turn to be a very large integer, one might be tempted to try out various values of n in the given relation, starting from 1 onwards, and see which one fits. This trial-and-error approach might quickly give a result in this particular example, but what would we do it n was possibly larger?

The generally followed approach in such examples, where the integral can be characterised by positive integer n (called the order of the integral), is to express it in terms of a lower order integral. If we denote the n^{th} order integral by I_n, we should try to express I_n in terms of I_k where $k < n$. Such a relation is called a recursive relation. We can then simply use this relation repeatedly on any order and obtain the integral of the next order (instead of every time repeating the calculation of integration again)

We will now use this approach on the current example:

Let $$I_n = \int_0^1 e^x (x-1)^n \, dx$$

To simplify I_n, first of all let $x - 1 = t \Rightarrow dx = dt$ and the limits become –1 to 0.

Thus,

$$I_n = \int_{-1}^{0} e^{t+1} t^n dt$$

$$= e \int_{-1}^{0} e^t \cdot t^n \; dt$$

We now use integration by parts to solve this integral, taking t^n as the first function:

$$I_n = e\left\{ t^n \cdot e^t \Big|_{-1}^{0} - n \int_{-1}^{0} \left(t^{n-1} \cdot \int e^t dt \right) dt \right\}$$

$$= e\left\{ \frac{(-1)^{n+1}}{e} - n \int_{-1}^{0} t^{n-1} \cdot e^t dt \right.$$

$$= (-1)^{n+1} - nI_{n-1} \qquad \ldots(1)$$

We have thus established the relation between I_n and I_{n-1} in (1)

Now observe that I_0 can easily be evaluated:

$$I_0 = e \int_{-1}^{0} e^t dt$$

$$= e - 1$$

Using (1) repeatedly, we can now obtain all the higher order integrals:

$$I_1 = (-1)^{1+1} - 1.I_0 = 2 - e$$
$$I_2 = (-1)^{2+1} - 2.I_1 = -5 + 2e$$
$$I_3 = (-1)^{3+1} - 3.I_2 = 16 - 6e$$

$n = 3$ is therefore the positive integer we had set out to determine. Notice the power of the recursive relation that we obtained in (1). Using that relation, it was just a matter of minor calculations to successively determine I_1, I_2 and I_3 from I_0. Without (1), we would have to apply integration by parts *everytime*, had we used the trial-and-error approach.

Lets look at another example of this sort. ◀

Example – 9

Evaluate $\int_0^\infty e^{-x} x^n dx$

Solution: Notice that no matter what n be,

$$\lim_{x \to \infty} \left(e^{-x} x^n \right) = 0$$

so that we will obtain a finite area under the curve.

Let $$I_n = \int_0^\infty e^{-x} x^n dx$$

We apply integration by parts on I_n:

$$I_n = -x^n e^{-x} \Big|_0^\infty + n \int_0^\infty x^{n-1} e^{-x} dx$$
$$= 0 + nI_{n-1}$$

Thus, our recursive relation is

$$I_n = nI_{n-1}$$

We use this repeatedly now:

$$\begin{aligned} I_n &= nI_{n-1} \\ &= n(n-1)I_{n-2} \\ &= n(n-1)(n-2)I_{n-3} \\ &\ddots \\ &= n!I_0 \end{aligned}$$

I_0 is simple to determine:

$$\begin{aligned} I_0 &= \int_0^\infty e^{-x} dx \\ &= -e^{-x} \Big|_0^\infty \\ &= 1 \end{aligned}$$

Thus, $$I_n = n!$$

Example – 10

Prove that $\int_0^{\pi} \frac{\sin(2n+1)x}{\sin x}dx = \pi$ for all $n \in \mathbb{N}$

Solution: Let $I_n = \int_0^{\pi} \frac{\sin(2n+1)x}{\sin x}dx.$

What is I_1?

$$I_1 = \int_0^{\pi} \frac{\sin 3x}{\sin x}dx$$

$$= \int_0^{\pi} (3 - 4\sin^2 x)dx$$

$$= \int_0^{\pi} (3 - 2(1-\cos 2x))dx$$

$$= (x + \sin 2x)\Big|_0^{\pi}$$

$$= \pi$$

Thus, $n = 1$ satisfies the stated property. How do we approach the general case?

Some reflection on the nature of the integral will show you that evaluating I_n by itself would be tedius. What we could instead do is this:

We have already shown that $I_1 = \pi.$ If we show that $I_{n+1} - I_n = 0,$ our task would be accomplished, since then I_2, I_3 and all the higher order integrals become equal to I_1, which is $\pi;$ this is what we want to prove.

$$I_{n+1} - I_n = \int_0^{\pi} \left\{ \frac{\sin(2n+3)x}{\sin x} - \frac{\sin(2n+1)x}{\sin x} \right\} dx$$

$$= 2\int_0^{\pi} \frac{\cancel{\sin x}\cos(2n+2)x}{\cancel{\sin x}}dx$$

$$= \frac{1}{n+1}\{\sin(2n+2)x\}\Big|_0^{\pi}$$

$$= 0$$

Therefore, $I_n = \pi \quad \forall\ n \in N$ ◀

Example – 11

Evaluate $\int_0^{\pi/2} \ln(\tan x)\,dx$

Solution: Careful observation will show that the function $\ln(\tan x)$ is odd about $x = \frac{\pi}{4}$, because,

$$\ln\left(\tan\left(\frac{\pi}{2} - x\right)\right) = \ln(\cot x) = \ln\left(\frac{1}{\tan x}\right) = -\ln(\tan x)$$

Thus, as discussed in property –7, the given integral will become 0. ◀

Example – 12

Evaluate $\int_0^2 \left[x^2 - 1\right] dx.$

Solution: The function to be integrated as been sketched below in the region of interest:

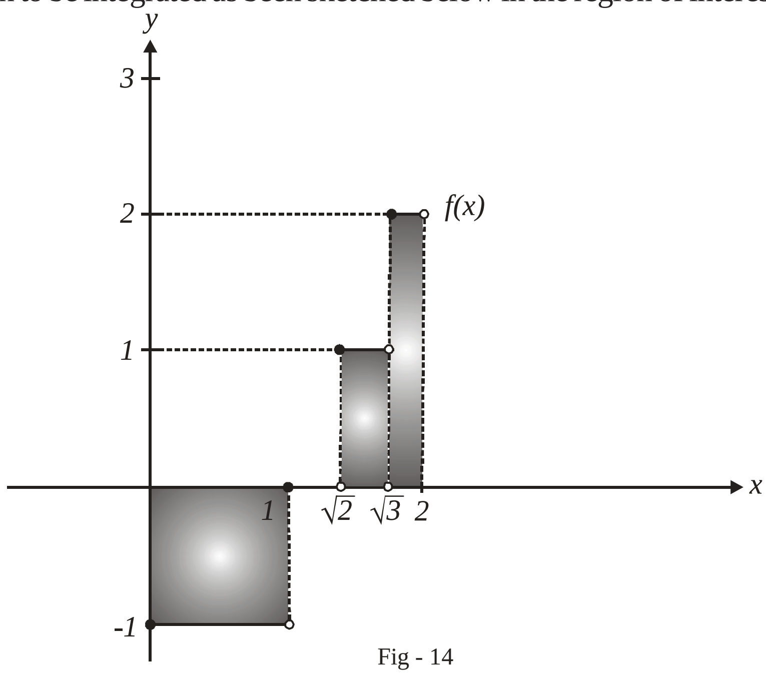


Fig - 14

This function is discontinuous; as discussed in property-3, we can split the required interval of integration. We will do it in such a way so that in each of the sub-intervals that we obtain, the function is continuous and can be integrated.

Thus, if $\left[x^2 - 1\right] = f(x)$, then

$$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^{\sqrt{2}} f(x)\,dx + \int_{\sqrt{2}}^{\sqrt{3}} f(x)\,dx + \int_{\sqrt{3}}^2 f(x)\,dx$$

$$= \int_0^1 (-1)\,dx + \int_1^{\sqrt{2}} (0)\,dx + \int_{\sqrt{2}}^{\sqrt{3}} (1)\,dx + \int_{\sqrt{3}}^2 (2)\,dx$$

$$= -1 + 0 + \left(\sqrt{3} - \sqrt{2}\right) + 2\cdot\left(2 - \sqrt{3}\right)$$

$$= 3 - \left(\sqrt{2} + \sqrt{3}\right)$$

In general, for any discontinuous function $f(x)$ whose integral we need to evaluate, the approach described above is followed. $f(x)$ is separately integrated in sub–intervals where it is continuous and the results so obtained are added. ◀

TRY YOURSELF - I

Q. 1 Prove that $\int_0^{\pi} \frac{\sin 2nx}{\sin x} dx = 0 \quad \forall\ n \in \mathbb{N}$

Q. 2 If n is an odd positive integer, prove that

$$I_n = \int_0^{\pi/2} \frac{\sin nx}{\sin x} dx = \frac{\pi}{2}$$

Q. 3 Show that $\int_0^{\pi/4} (\cos 2\theta)^{3/2} \cos\theta \, d\theta = \frac{3\pi}{16\sqrt{2}}$

Q. 4 Show that $\int_0^{\pi/3} \frac{x}{1+\sec x} dx = \frac{\pi^2}{18} - \frac{\pi}{3\sqrt{3}} + 2\ln\frac{2}{\sqrt{3}}$

Q. 5 For $x > 0$, if $f(x) = \int_1^{x} \frac{\ln z}{z^2 + z + 1} dz$, show that $f(x) = f\left(\frac{1}{x}\right)$

Q. 6 If $I_n = \int_0^{\pi} \frac{1-\cos nx}{1-\cos x} dx$ where n is a non-negative integer, show that I_n, I_{n+1} and I_{n+2} are in *A.P.*

Q. 7 Show that $\int_a^{b} \frac{1}{\sqrt{(x-a)(b-x)}} dx,\ b > a$ is equal to π .

Q. 8 If $I_n = \int_0^{\pi/2} \cos^n x \, dx, n \in N,$ show that $I_n = \frac{n-1}{n} I_{n-2}$

Q. 9 Prove that $\int_0^{\pi} \frac{1}{3+2\sin x + \cos x} dx = \frac{\pi}{4}$

Q. 10 Prove that $\int_2^{3} \frac{2x^5 + x^4 - 2x^3 + 2x^2 + 1}{(x^2+1)(x^4-1)} = \frac{\ln 6}{2} - \frac{1}{10}$

* You must have used substitutions in some of the questions above. Think about the validity of these substitutions. Is a substitution always valid? Or do we need to fulfill certain requirements if a substitution is to be valid?