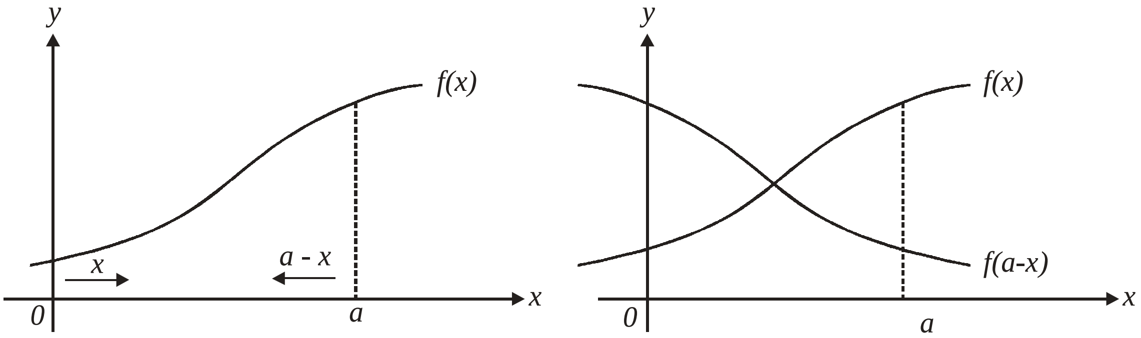
Section - 2 MORE PROPERTIES

In section -1 , we dealt with some basic properties that definite integrals satisfy. This section continues with the development of some more properties that are not so trivial, and, when properly applied, can be extremely powerful. However, even the justifications for these properties stem from obvious physical (graphical) interpretations of functions.

(9) $$\int_0^a f(x)\,dx = \int_0^a f(a-x)dx$$

This property says that when integrating from 0 to a, we will get the same result whether we use the function $f(x)$ or $f(a-x)$. The justification for this property will become clear from the figures below:



As x progresses from 0 to a,
the variable a - x progresses from a to 0.
Thus, whether we use x or a - x,
the entire interval [0, a] is still covered.

The function f(a - x) can be obtained from the function f(x) by first flipping f(x) along the y-axis and then shifting it right by a units. Notice that in the interval [0, a], f(x) and f(a - x) describe precisely the same area.

Fig - 15

There are two ways to look at the justification of this property, as described in the figures on the left and right respectively.

Let us see how to apply this property usefully:

Example – 13

Evaluate $I = \displaystyle\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$.

Solution: Observe that evaluating the indefinite integral of the function above would be very tedious. Using Property-9, this integral can be immediately simplified; we use the substitution $x \to \dfrac{\pi}{2} - x$.

Therefore, I becomes:

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$$

Now, we add the original expression for I and this newly obtained modified expression:

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$= \int_0^{\pi/2} dx \quad \text{(Quite simple now!)}$$

$$= \frac{\pi}{2}$$

$$\Rightarrow \; I = \frac{\pi}{4}$$

◀

Example – 14

Evaluate $I = \int_0^{\pi/4} \ln(1+\tan x)\,dx$

Solution: We again use property -9 and try to simplify this integral; use the substitution $x \to \frac{\pi}{4} - x$ in the function to be integrated:

$$I = \int_0^{\pi/4} \ln\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$$

$$= \int_0^{\pi/4} \ln\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$$

$$= \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan x}\right)dx$$

$$= \int_0^{\pi/4} \{\ln 2 - \ln(1+\tan x)\}\,dx$$

$$= \int_0^{\pi/4} \ln 2\;dx - I$$

$$= \frac{\pi}{4}\ln 2 - I$$

$$\Rightarrow \; 2I = \frac{\pi}{4}\ln 2$$

$$\Rightarrow \; I = \frac{\pi}{8}\ln 2$$

Again, we see that using property - 9 saved us from the tedious task of first evaluating the anti-derivative.

◀

Example – 15

Evaluate $I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$

Solution: We first use property -9 to simplify this integral; use the substitution $x \to \frac{\pi}{2} - x$ in the function to be integrated:

$$I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) \cos x \sin x}{\cos^4 x + \sin^4 x} dx$$

Adding the original and the modified expressions of I, we obtain :

$$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

$$\Rightarrow \; I = \frac{\pi}{4} \int_0^{\pi/2} \frac{\tan x \sec^2 x}{1 + \tan^4 x} dx$$

Now we use the substitution $\tan^2 x = y$

$$\Rightarrow \; 2 \tan x \sec^2 x \; dx = dy$$

and when $x = 0 \Rightarrow y = 0$

$$x = \frac{\pi}{2} \Rightarrow y = \infty$$

I can now be written in terms of y as :

$$I = \frac{\pi}{8} \int_0^{\infty} \frac{dy}{1 + y^2}$$

$$= \frac{\pi}{8} \left(\tan^{-1} y\right)\Big|_0^{\infty}$$

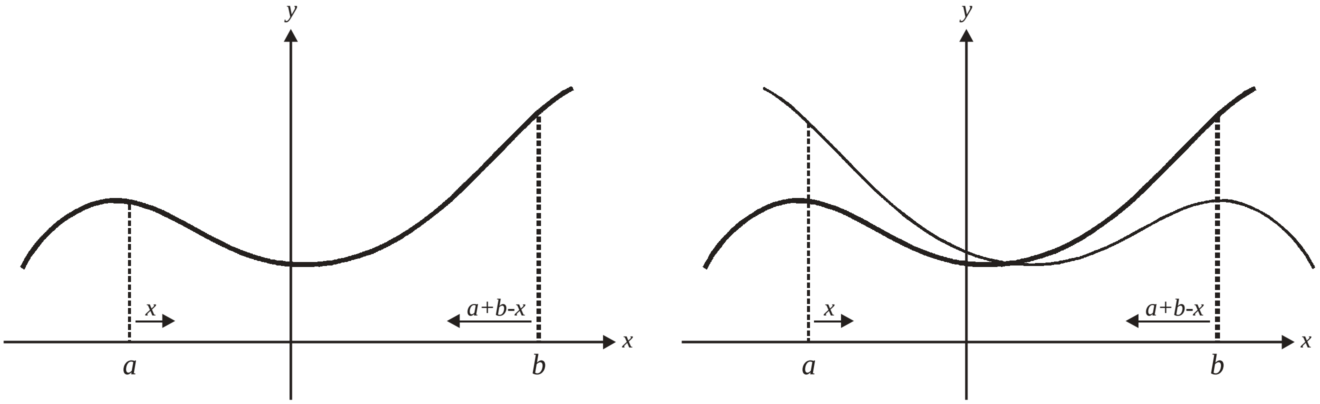
$$= \frac{\pi^2}{16}$$

◀

(10) We can now extend property -9 into a more general property :

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

Graphically, the justification for this property is analogous to that for property -9



As the variable x varies from a to b,
the variable a + b - x varies from b to a.
Thus, whether we use x or a + b - x,
the entire interval [a, b] is covered in both
the cases and the areas will be the same

The graph of f(a+b-x) can be obtained from the graph of f(x) by first flipping the graph of f(x) along the y-axis and then shifting it (a+b) units towards the right; the areas described by f(x) and f(a+b-x) in the interval [a, b] are precisely the same

Fig - 16

There is a straight forward analytical justification also. Use $a+b-x=t$ in the right side integral for that purpose. Property -10 is one of the most widely used properties to simplify definite integrals.

Example – 16

Evaluate $I = \int\limits_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\tan x}}\,dx$

Solution:

$$I = \int\limits_{\pi/6}^{\pi/3} \frac{1}{1+\frac{\sqrt{\sin x}}{\sqrt{\cos x}}}\,dx$$

$$= \int\limits_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx \qquad \text{...(1)}$$

We now use property -10; we substitute $x \to \left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)$ i.e. $\left(\frac{\pi}{2}-x\right)$ in the function to be integrated. Thus, I becomes:

$$I = \int\limits_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx \qquad \text{...(2)}$$

Adding (1) and (2), we obtain

$$2I = \int\limits_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$= \int_{\pi/6}^{\pi/3} dx$$

$$= \frac{\pi}{6}$$

$$\Rightarrow \; I = \frac{\pi}{12}$$ ◀

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(11) $\int_{0}^{2a} f(x)dx = \int_{0}^{a} \{f(x) + f(2a-x)\}dx$

The justification for this property is described below:

$$\int_{0}^{2a} f(x)dx = \int_{0}^{a} f(x)dx + \int_{a}^{2a} f(x)dx.$$

To evaluate $\int_{a}^{2a} f(x)dx$, we can equivalently use the variable $(2a-x)$ instead of x, but the limits of integration will change from (a to $2a$) to (0 to a). This is because as x varies from 0 to a, $2a-x$ will vary from ($2a$ to a) covering the same interval [a, $2a$]. Thus,

$$\int_{a}^{2a} f(x)dx = \int_{0}^{a} f(2a-x)\,dx$$

Hence, the stated assertion is valid

Example – 17

If f is an even function, then prove that $\int_{0}^{\pi/2} f(\cos 2x)\cos x\,dx = \sqrt{2}\int_{0}^{\pi/4} f(\sin 2x)\cos x\,dx$

Solution: On the left side, the integration limits are $\left(0 \text{ to } \frac{\pi}{2}\right)$ while on the right side, they are $\left(0 \text{ to } \frac{\pi}{4}\right)$. Thus, it would be appropriate to use Property -11

$$\int_{0}^{\pi/2} f(\cos 2x)\cos x\,dx = \int_{0}^{\pi/4} \left\{ f(\cos 2x)\cos x + f\left(\cos 2\left(\frac{\pi}{2}-x\right)\right)\cos\left(\frac{\pi}{2}-x\right)\right\}dx$$

$$= \int_{0}^{\pi/4} \{f(\cos 2x)\cos x + f(-\cos 2x)\sin x\}\,dx$$

[since f is an even function, $f(-\cos 2x) = f(\cos 2x)$]

$$= \int_{0}^{\pi/4} f(\cos 2x)\{\cos x + \sin x\}\,dx$$

$$= \sqrt{2}\int_{0}^{\pi/4} f(\cos 2x)\sin\left(x+\frac{\pi}{4}\right)dx \qquad ...(1)$$

$$\left[\begin{array}{l}\text{Now we use property - 9 to obtain the final form that we require;}\\ \text{use the substitution } x \to \dfrac{\pi}{4} - x \text{ in the function to be integrated in (1)}\end{array}\right]$$

$$= \sqrt{2}\int_0^{\pi/4} f\left(\cos 2\left(\frac{\pi}{4} - x\right)\right)\sin\left(\frac{\pi}{4} - x + \frac{\pi}{4}\right)dx$$

$$= \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\cos x\ dx$$

◀

We could also have started with property -9 directly:

$$I = \int_0^{\pi/2} f(\cos 2x)\cos x\ dx \qquad \text{...(2)}$$

$$= \int_0^{\pi/2} f\left(\cos 2\left(\frac{\pi}{2} - x\right)\right)\cos\left(\frac{\pi}{2} - x\right)dx$$

$$= \int_0^{\pi/2} f\left(-\cos 2x\right)\sin x\,dx$$

$$= \int_0^{\pi/2} f\left(\cos 2x\right)\sin x\,dx \qquad \text{...(3)}$$

Adding (2) and (3) we obtain

$$2I = \int_0^{\pi/2} f(\cos 2x)(\sin x + \cos x)\,dx$$

Notice now that the function being integrated on the right side above is symmetric about $\frac{\pi}{4}$; i.e., if we substitute $\frac{\pi}{2} - x$ for x, we will obtain the same function again. Thus, (property 7):

$$2I = 2\int_0^{\pi/4} f(\cos 2x)(\sin x + \cos x)\,dx$$

$$= 2\sqrt{2}\int_0^{\pi/4} f(\cos 2x)\sin\left(x + \frac{\pi}{4}\right)dx$$

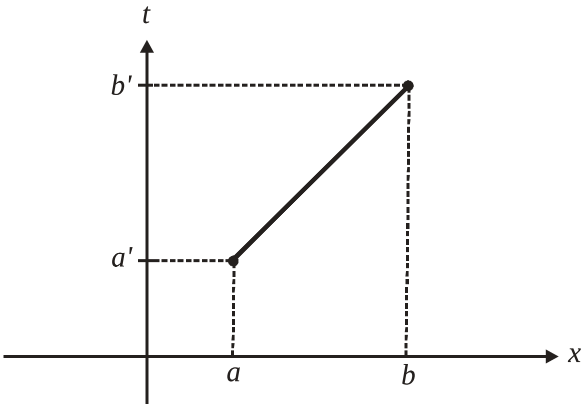
$$\Rightarrow\ I = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\sin\left(x + \frac{\pi}{4}\right)dx$$

This is the same expression that we had obtained in (1). From here, we can proceed as described earlier.

◀

(12) Sometimes, it is convenient to change the limits of integration into some other limits. For example, suppose we have to add two definite integrals I_1 and I_2; the limits of integration for these integrals are different. If we could somehow change the limits of I_2 into those of I_1 or vice-versa, or in fact change the limits of both I_1 and I_2 into a third (common) set of limits, the addition could be accomplished easily.

Suppose that $I = \int_a^b f(x)\,dx$. We need to change the limits (*a* to *b*) to (*a*' to *b*'). As *x* varies from *a* to *b*, we need a new variable *t* (in terms of *x*) which varies from *a*' to *b*'.



As x varies from a to b, t varies from a' to b'.
Thus,

$$\frac{t - a'}{x - a} = \frac{b' - a'}{b - a}$$

Fig - 17

As described in the figure above, the new variable *t* is given by,

$$t = a' + \left(\frac{b' - a'}{b - a} \right)(x - a) \; .$$

Thus,

$$dt = \frac{b' - a'}{b - a} dx$$

$$\Rightarrow \; I = \int_a^b f(x)\,dx$$

$$= \int_{a'}^{b'} f\left(a + \left(\frac{b - a}{b' - a'} \right)(t - a') \right)\left(\frac{b - a}{b' - a'} \right) dt$$

The modified integral has the limits $(a' \text{ to } b')$. A particular case of this property is modifying the arbitrary integration limits (*a* to *b*) to (0 to 1) i.e., $a' = 0$ and $b' = 1$. For this case,

$$I = \int_a^b f(x)\,dx$$

$$= (b - a)\int_0^1 f(a + (b - a)t)\,dt$$

(13) If $f(x)$ is a periodic function with period T, then the area under $f(x)$ for n periods would be n times the area under $f(x)$ for one period, i.e.

$$\int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx$$

Now, consider the periodic function $f(x) = \sin x$ as an example. The period of $\sin x$ is 2π .

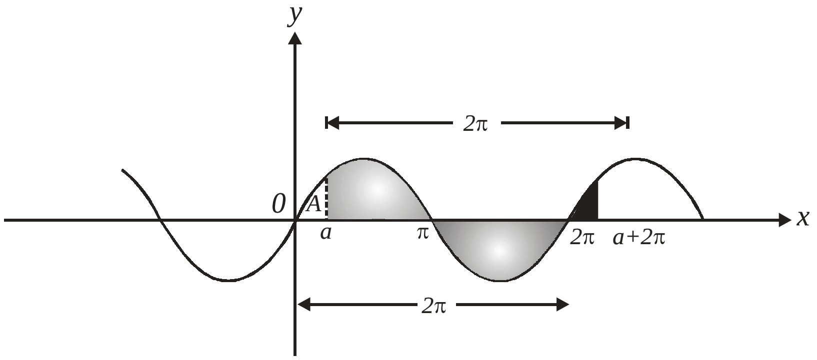


Fig - 18

Suppose we intend to calculate $\int_a^{a+2\pi} \sin x\,dx$ as depicted above. Notice that the darkly shaded area in the interval $[2\pi,\ a+2\pi]$ can precisely cover the area marked as A. Thus,

$$\int_a^{a+2\pi} \sin x\,dx = \int_0^{2\pi} \sin x\ dx$$

This will hold true for every periodic function, i.e.

$$\int_a^{a+T} f(x)\,dx = \int_0^{T} f(x)\,dx \qquad \text{(where } T \text{ is the period of } f(x))$$

This also implies that

$$\int_a^{a+nT} f(x)\,dx = \int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx$$

and $$\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^{b} f(x)\,dx$$

and $$\int_a^{b+nT} f(x)\,dx = \int_a^{b} f(x)\,dx + n\int_0^{T} f(x)\,dx$$

Example – 18

Show that $\int_{0}^{n\pi+V} |\sin x|\,dx = (2n+1)-\cos V$, where n is a positive integer and $V = [0,\pi)$

Solution: $f(x) = |\sin x|$ is periodic with period π.

Therefore, as described in property-13,

$$\int_{0}^{n\pi+V} |\sin x|\,dx = \int_{0}^{V} |\sin x|\,dx + n\int_{0}^{\pi} |\sin x|\,dx$$

$$= \int_{0}^{V} \sin x dx + n\int_{0}^{\pi} \sin x dx \qquad \left[\begin{array}{l}\because\ \sin x \ge 0 \\ \quad \text{for } x \in [0,\pi)\end{array}\right]$$

$$= -\cos x\Big|_{0}^{V} + n(-\cos x)\Big|_{0}^{\pi}$$

$$= 1-\cos V + n(2)$$

$$= (2n+1)-\cos V$$

◀

Example – 19

Evaluate $\int_{0}^{n\pi} \sqrt{1-\cos 2x}\,dx$, where n is a positive integer.

Solution:

$$I = \int_{0}^{n\pi} \sqrt{1-\cos 2x}\,dx$$

$$= \int_{0}^{n\pi} \sqrt{2\sin^2 x}\,dx$$

$$= \sqrt{2}\int_{0}^{n\pi} |\sin x|\,dx$$

$$= \sqrt{2}n\int_{0}^{\pi} |\sin x|\,dx \qquad \text{(Property-13)}$$

$$= 2\sqrt{2}n$$

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14. In the unit on “Integration Basics”, we saw that for a function $f(x)$, the anti-derivative $g(x)$ was defined as

$$g(x) = \int_{a}^{x} f(t)dt \qquad \text{(where } a \text{ is a constant)}$$

so that $g'(x) = f(x)$

We now consider an integral of the following form:

$$h(x) = \int_{\phi(x)}^{\psi(x)} f(t)dt$$

That is, the limits of integration are themselves functions of *x*. The anti-derivative $g(x)$ is a special case of $h(x)$ with $\psi(x) = x$ and $\phi(x) = a.$

Now, how do we evaluate $h'(x)$? Leibnitz's rule for differentiation tells us how to do so. Since $g(x)$ is the anti-derivative of $f(x),$ we have:

$$\begin{aligned} h(x) &= \int_{\phi(x)}^{\psi(x)} f(t)dt \\ &= g(t)\Big|_{\phi(x)}^{\psi(x)} \\ &= g\left(\psi(x)\right) - g\left(\phi(x)\right) \\ \Rightarrow \quad h'(x) &= g'\left(\psi(x)\right)\psi'(x) - g'\left(\phi(x)\right)\phi'(x) \\ &= f\left(\psi(x)\right)\psi'(x) - f\left(\phi(x)\right)\phi'(x). \end{aligned}$$

Let us see an example of this rule:

Example – 20

Evaluate $f'(x)$ if $f(x) = \int_{x}^{x^2} \left(t^2+1\right)dt$

Solution: Let us first find out $f(x)$ using straight forward integration:

$$\begin{aligned} f(x) &= \left(\frac{t^3}{3} + t\right)\Bigg|_{x}^{x^2} \\ &= \frac{x^6}{3} + x^2 - \frac{x^3}{3} - x \\ \Rightarrow \quad f'(x) &= 2x^5 - x^2 + 2x - 1 \end{aligned}$$

Now we redo this using the Leibnitz's differentiation rule:

$$\begin{aligned} f'(x) &= \left(\left(x^2\right)^2 + 1\right)\left(x^2\right)' - \left(\left(x\right)^2 + 1\right)\left(x\right)' \\ &= (x^4+1)(2x) - (x^2+1)(1) \\ &= 2x^5 - x^2 + 2x - 1 \end{aligned}$$

◀

Let us now do an example of this rule where straight forward integration would be much more difficult.

Example – 21

Determine the equation of the tangent to the curve $y = f(x)$ at $x = 1$, where

$$f(x) = \int_{x^2}^{x^3} \frac{1}{\sqrt{1+t^5}}\,dt$$

Solution: Notice how tedious it would be to actually carry out the integration. Instead, we use the Leibnitz's differentiation rule:

$$f'(x) = \frac{3x^2}{\sqrt{1+x^{15}}} - \frac{2x}{\sqrt{1+x^{10}}}$$

$$\Rightarrow \qquad f'(1) = \frac{3}{\sqrt{2}} - \frac{2}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}$$

Also, at $x = 1$,

$$f(x) = f(1) = \int_1^1 \frac{1}{\sqrt{1+t^5}}\,dt$$

$$= 0$$

Thus, the tangent passes through (1, 0) and has slope $\frac{1}{\sqrt{2}}$. The required equation is

$$y - 0 = \frac{1}{\sqrt{2}}(x-1)$$

$$\Rightarrow \qquad x - \sqrt{2}y - 1 = 0$$

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(15). Consider a function in two variables x and y, i.e.,

$$z = f(x, y)$$

Let us consider the integral of z with respect to x, from a to b, i.e.,

$$I = \int_a^b f(x, y)dx$$

For this integration, the variable is only x and not y. y is essentially a constant for the integration process. Therefore, after we have evaluated the definite integral and put in the integration limits, y will still remain in the expression of I. This means that I is a function of y.

$$\Rightarrow \qquad I(y) = \int_a^b f(x, y)dx \qquad \text{...(1)}$$

Our 15th property says that the relation (1) can be differentiated with respect to y as follows:

$$I'(y) = \frac{d}{dy}\left(\int_a^b f(x,y)\,dx\right)$$

$$= \int_a^b \frac{\partial f(x,y)}{\partial y}\,dx$$

where $\frac{\partial f(x,y)}{\partial y}$ stands for the partial derivative of $f(x,y)$ with respect to y, that is , the derivative of $f(x,y)$ w.r.t. y, treating x as a constant

Let us see the justification for this property:

$$I'(y) = \lim_{h\to 0}\frac{I(y+h)-I(y)}{h}$$

$$= \lim_{h\to 0}\frac{\int_a^b f(x,y+h)dx - \int_a^b f(x,y)dx}{h}$$

$$= \int_a^b \left\{\lim_{h\to 0}\left(\frac{f(x,y+h)-f(x,y)}{h}\right)\right\}dx$$

$$= \int_a^b \frac{\partial f(x,y)}{\partial y}\,dx$$

This property turns out to be very useful in certain cases.

Example – 22

Evaluate $I = \int_0^1 \frac{x^k - 1}{\ln x}\,dx$

Solution: Observe that I will be a function of k. Instead of carrying out direct integration, we use property -15:

$$\frac{dI(k)}{dk} = \int_0^1 \frac{\partial}{\partial k}\left(\frac{x^k-1}{\ln x}\right)dx$$

$$= \int_0^1 \frac{x^k \ln x}{\ln x}\,dx$$

$$= \int_0^1 x^k\,dx$$

$$= \frac{1}{k+1}.$$

Thus,

$$dI(k) = \frac{dk}{k+1}$$

Integrating both sides, we obtain

$$I(k) = \ln(k+1) + C \qquad \text{...(1)}$$

To obtain *C*, note from the original definition of *I* that *I* (0) = 0. Using this in (1), we obtain

$$0 = \ln 1 + C$$

$$\Rightarrow \qquad C = 0$$

Thus,

$$I(k) = \ln(k+1)$$ ◀

Observe again carefully the indirect route that property-15 offered us to solve this integral.

Example – 23

Evaluate $I = \int_0^1 \frac{x^\alpha - x^\beta}{\ln x} dx$

Solution: We again try to use property -15 to solve this integral. Let us treat *I* as a function of α. Therefore,

$$I(\alpha) = \int_0^1 \frac{x^\alpha - x^\beta}{\ln x} dx$$

Notice that

$$I(\beta) = \int_0^1 \frac{x^\beta - x^\beta}{\ln x} dx = 0$$

Now, using property-15 we obtain:

$$\frac{dI(\alpha)}{d\alpha} = \int_0^1 \frac{\partial}{\partial \alpha}\left(\frac{x^\alpha - x^\beta}{\ln x}\right) dx$$

$$= \int_0^1 \frac{x^\alpha \ln x}{\ln x} dx$$

$$= \frac{1}{\alpha + 1}$$

Thus,

$$dI(\alpha) = \frac{d\alpha}{\alpha+1}$$

$$\Rightarrow \qquad I(\alpha) = \ln(\alpha+1) + C$$

Using $I(\beta) = 0$ above, we obtain

$$I(\beta) = 0 = \ln(\beta+1) + C$$

$$\Rightarrow \qquad C = 0 - \ln(\beta+1) = -\ln\left(\beta+1\right)$$

Thus,

$$I(\alpha) = \ln(\alpha+1) - \ln(\beta+1)$$

$$= \ln\left(\frac{\alpha+1}{\beta+1}\right)$$

This is the required integral ! ◀