Section - 3 INTEGRATION AS LIMIT OF A SUM

In this section, we will very briefly revisit the discussion we had in the unit "Integration Basics" on viewing integration as a limit of a sum. If you do not remember that discussion, you are advised to refer to it again.

In that discussion, we saw that the area under the curve is the sum of an infinitely large number of rectangles with infinitesimally small widths. These rectangles precisely give the area under the curve in the limit that their number tends to infinity, i.e.,:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{r=1}^{n} hf(a+rh)$$

where $$h = \frac{b-a}{n}$$

In particular, notice that if the lower limit a is 0, then

$$\int_0^b f(x)dx = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{b}{n} f\left(\frac{br}{n}\right) \qquad ...(1)$$

We sometimes encounter series of the form as in the right hand side of (1). The discussion above shows that such series can be summed using definite integrals.

This will become more clear through examples.

Example – 24

(a) Find the sum of the series

$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{3n} \quad \text{as } n \to \infty$$

(b) Find the sum of the series

$$\frac{n}{n^2+1^2} + \frac{1}{n^2+2^2} + ... + \frac{1}{n^2+n^2} \quad \text{as } n \to \infty$$

(c) Evaluate

$$\lim_{n\to\infty} \frac{[(n+1)(n+2).....(n+n)]^{1/n}}{n}$$

Solution: **(a)** The given series can be written concisely as

$$S = \sum_{r=0}^{2n} \frac{1}{n+r}$$

$$= \frac{1}{n} \sum_{r=0}^{2n} \frac{1}{1+(r/n)}$$

Comparing this with the right hand side of (1), we see that S can be expressed as the integral of a function $f(x) = \frac{1}{1+x}$ from 0 to 2, because since r varies till $2n$, r/n varies till 2. Thus,

$$S = \int_0^2 \frac{1}{1+x} dx$$

$$= \ln 3$$

◄

(b) Concisely put,

$$S = \sum_{l=1}^{n} \frac{n}{n^2 + r^2}$$

$$= \frac{1}{n} \sum_{r=1}^{n} \frac{1}{1 + (r/n)^2}$$

As discussed earlier, S can be written in integral form as

$$S = \int_0^1 \frac{1}{1+x^2}\, dx$$

$$= \tan^{-1} x \Big|_0^1$$

$$= \frac{\pi}{4}$$ ◀

(c) Let L represent the given limit. We have,

$$\ln L = \frac{1}{n} \ln \left\{ \left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right) \ldots\ldots \left(1 + \frac{n}{n}\right) \right\}$$

$$= \frac{1}{n} \sum_{r=1}^{n} \ln \left(1 + \frac{r}{n}\right)$$

Thus,

$$\ln L = \int_0^1 \ln(1+x)\, dx$$

$$= x \ln(1+x)\Big|_0^1 - \int_0^1 \frac{x}{1+x}\, dx$$

$$= \ln 2 - \int_0^1 \left(\frac{x+1-1}{x+1} \right) dx$$

$$= \ln 2 - 1 + \ln 2$$

$$= 2 \ln 2 - 1$$

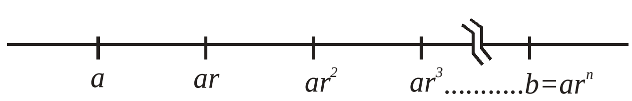
$$= \ln(4/e)$$

This implies that

$$L = \frac{4}{e}$$ ◀

From these three examples, the usefulness of definite integrals in summing series should be quite apparent.

One last thing about definite integration as the limit of a sum form: when we divide the area we want to evaluate into n rectangles, we need not have those n rectangles of the same width. The widths can be arbitrary as long as all of them tend to 0 in the limit $n \to \infty$. For example, we could divide the interval $[a, b]$ as follows:



To divide [a, b] into n intervals, we let $ar^n = b$.

Thus, $n = \dfrac{\ln (b/a)}{\ln r}$

As $r \to 1$, $n \to \infty$

Fig. 19

In the scheme of division above, as $r \to 1$, $n \to \infty$. This is also a valid division scheme since the width of the k^{th} interval is $ar^k - ar^{k-1}$ which tends to 0 as $r \to 1$ or $n \to \infty$.

In fact, sometimes a uniform division scheme might not work due to the nature of the function to be integrated. In such cases, the division scheme described above could be used.

Example – 25

Evaluate the integral $\int_a^b \frac{1}{x}dx$ using first principles.

Solution: As described earlier, we divide the interval $[a, b]$ into n non-uniform sub intervals, the k^{th} interval being of width $w_k = ar^k - ar^{k-1} = ar^{k-1}(r-1).$ The sum of the areas of the n rectangles is

$$S = \sum_{k=1}^{n} w_k \; f(ar^{k-1})$$

$$= \sum_{k=1}^{n} ar^{k-1}(r-1) \cdot \frac{1}{ar^{k-1}}$$

$$= \sum_{k=1}^{n} (r-1)$$

$$= n(r-1)$$

$$= \frac{(r-1)\ln(b/a)}{\ln r}$$

$$= \frac{p\ln(b/a)}{\ln(p+1)} \qquad \left(\begin{matrix}\text{where } p = r-1; \\ \text{as } r \to 1, p \to 0\end{matrix}\right)$$

Thus,

$$\lim_{n\to\infty} S = \lim_{r\to 1} S = \lim_{p\to 0} S = \ln(b/a)$$

$$\Rightarrow \qquad \int_a^b \frac{1}{x}dx = \ln\left(\frac{b}{a}\right)$$

◀

TRY YOURSELF - II

Q. 1 Prove that $\int_0^{\pi} \frac{x\sin x}{2-\sin^2 x}dx = \frac{\pi^2}{4}$

Q. 2 Prove that $\int_0^{\pi} \frac{x^2 \cos x}{(1+\sin x)^2}dx = \pi(2-\pi)$

Q. 3 Prove that $\int_0^{\pi} \ln(1+\cos x)dx = -\pi \ln 2$

Q. 4 Prove that $\int_0^{\pi} \frac{x}{1+\sin\alpha \sin x}dx = \frac{\pi}{\cos\alpha}\left(\frac{\pi}{2}-\alpha\right)$

Q. 5 If $y = \int_0^{x^2} \ln(1+x)dx,$ show that $\frac{dy}{dx} = 2x\ln(1+x^2)$

Q. 6 Prove that $\int_a^{b} \left([x]+[-x]\right)dx = a-b$

Q. 7 Show that $\int_0^{\pi/2} \sin 2x \ln(\tan x)dx = 0$

Q. 8 Show that $\int_0^{\pi} \frac{x}{1+\cos^2 x}dx = \frac{\pi^2}{2\sqrt{2}}$

Q. 9 If $I_n = \int_0^1 x^n \tan^{-1} x \; dx,$ prove that $(n+1)I_n + (n-1)I_{n-2} = \frac{\pi}{2} - \frac{1}{n}$

Q. 10 Evaluate $\lim_{n\to\infty} \frac{(n!)^{1/n}}{n}$

SOLVED EXAMPLES

Example – 1

Evaluate $\int_0^{n/2} \frac{x \sin x \cos x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} dx$ where $a, b > 0$

Solution: Consider only the function $f(x) = \frac{\sin x \cos x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2}$

Observe that $f(x)$ can be easily integrated. Using the substitution $\sin^2 x = t,$ we get the integral as (verify):

$$\int f(x) dx = \frac{-1}{2(b^2 - a^2)(a^2 \cos^2 x + b^2 \sin^2 x)}$$

Now since we know the integral of $f(x)$, we use integration by parts on the original integral, which is nothing but:

$$I = \int_0^{\pi/2} x\, f(x) dx$$

Applying integration by parts, we obtain:

$$I = \left(x \int f(x) dx \right) \Big|_0^{\pi/2} - \int_0^{\pi/2} \left(\int f(x)\, dx \right) dx$$

$$= \frac{1}{2(b^2 - a^2)} \left\{ \frac{-x}{a^2 \cos^2 x + b^2 \sin^2 x} \Bigg|_0^{\frac{\pi}{2}} + \int_0^{\pi/2} \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x} dx \right\}$$

$$= \frac{1}{2(b^2 - a^2)} \left\{ \frac{-\pi}{2b^2} + \int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2 \tan^2 x} dx \right\}$$

$\downarrow$

{To evaluate this integral, use the substitution $\tan x = t;$ this integral reduces to $\int_0^{\infty} \frac{dt}{a^2 + b^2 t^2}$ which is $\frac{\pi}{2ab}$}

$$= \frac{1}{2(b^2 - a^2)} \left\{ \frac{\pi}{2ab} - \frac{\pi}{2b^2} \right\}$$

$$= \frac{\pi}{4ab^2(a+b)}.$$

◀

Example – 2

Evaluate $I = \int_0^1 \ln\left(\sqrt{1-x} + \sqrt{1+x}\right) dx$

Solution: The expression for I contains both $(1-x)$ and $(1+x)$; the substitution $x=\cos 2\theta$ could do the job.

$$x=\cos 2\theta$$

$$\Rightarrow \quad dx=-2\sin 2\theta d\theta$$

$$\Rightarrow \quad \text{when } x=0,\ \theta=\pi/4$$

$$\text{when } x=1,\ \theta=0.$$

Thus, I gets modified to

$$I=-2\int_{\pi/4}^{0} \ln\left(\sqrt{1-\cos 2\theta}+\sqrt{1+\cos 2\theta}\right)\sin 2\theta d\theta$$

$$=2\int_0^{\pi/4} \ln\left\{\sqrt{2}\,(\sin\theta+\cos\theta)\right\}\sin 2\theta d\theta$$

$$=2\int_0^{\pi/4}\left(\ln\sqrt{2}\right)\sin 2\theta d\theta+2\int_0^{\pi/4}\ln(\underbrace{\sin\theta+\cos\theta}_{\text{First Function}})\underbrace{\sin 2\theta}_{\text{Second Function}}\, d\theta$$

$$=-\ln\sqrt{2}(\cos 2\theta)\Big|_0^{\pi/4}+2\left\{\frac{-\ln(\sin\theta+\cos\theta)\cos 2\theta}{2}\Bigg|_0^{\pi/4}+\int_0^{\pi/4}\frac{\cos\theta-\sin\theta}{\sin\theta+\cos\theta}\cdot\frac{\cos 2\theta}{2}\,d\theta\right\}$$

$$=\ln\sqrt{2}+2\left\{0+\frac{1}{2}\int_0^{\pi/4}(\cos\theta-\sin\theta)^2\,d\theta\right\}$$

$$=\ln\sqrt{2}+\int_0^{\pi/4}(\cos\theta-\sin\theta)^2 d\theta$$

$$=\ln\sqrt{2}+\int_0^{\pi/4}(1-\sin 2\theta)d\theta$$

$$=\ln\sqrt{2}+\frac{\pi}{4}+\frac{\cos 2\theta}{2}\Bigg|_0^{\pi/4}$$

$$=\ln\sqrt{2}+\frac{\pi}{4}-\frac{1}{2}$$

◀

Example – 3

If $m,n\in\mathbb{N}$, prove that $I_{m,n}=\int_0^1 x^m(1-x)^n\,dx=\dfrac{m!n!}{(m+n+1)!}$

Solution: It is clear from the final result which we need to obtain that we require a recursive relation involving $I_{m,n}$ and a lower order integral (w.r.t one of the two variables m or n).

$$I_{m,n} = \int_0^1 \underset{\substack{\uparrow \\ \text{Ist} \\ \text{Function}}}{(1-x)^n} \underset{\substack{\text{IInd} \\ \text{Function}}}{x^m} \; dx$$

$$= \left.\frac{(1-x)^n x^{m+1}}{m+1}\right|_0^1 + \frac{n}{m+1}\int_0^1 (1-x)^{n-1} x^{m+1} dx$$

$$= 0 + \frac{n}{m+1} I_{m+1,n-1}$$

Thus, the required recursive relation is

$$I_{m,n} = \frac{n}{m+1} I_{m+1,n-1}$$

We use this relation repeatedly now till n reduces to 0:

$$I_{m,n} = \frac{n}{m+1} I_{m+1,n-1}$$

$$= \frac{n(n-1)}{(m+1)(m+2)} I_{m+2,n-2}$$

$$\vdots$$

$$= \frac{n(n-1)............1}{(m+1)(m+2)....(m+n)} I_{m+n,0}$$

$$= \frac{m!n!}{(m+n)!} I_{m+n,0}$$

$I_{m+n,0}$ is easy to evaluate. Verify that it is $\dfrac{1}{m+n+1}$

Thus, $$I_{m,n} = \frac{m!n!}{(m+n+1)!}$$ ◀

Example – 4

Evaluate $I = \int_0^1 \cot^{-1}(1+x^2-x)dx$

Solution: Before reading the solution below, you are advised to try simplifying I directly using the properties and techniques developed in this chapter. You'll realise the difficulty of the task.

In this example, instead of direct application of any property, a non-trivial manipulation is first required to simplify I before applying any properties. That manipulation is now described:

$$I = \int_0^1 \cot^{-1}(1 + x^2 - x)dx$$

$$= \int_0^1 \tan^{-1}\left(\frac{1}{1 + x(x-1)}\right)dx$$

$$= \int_0^1 \tan^{-1}\left(\frac{x-(x-1)}{1 + x(x-1)}\right)dx \qquad \left\{\begin{matrix}\text{We wrote the numerator '1'} \\ \text{as '}x-(x-1)\text{'}\end{matrix}\right\}$$

$$= \int_0^1 \left\{\tan^{-1}x - \tan^{-1}(x-1)\right\}dx \qquad \left\{\because \ \tan^{-1}\left(\frac{A-B}{1+AB}\right) = \tan^{-1}A - \tan^{-1}B\right\}$$

$$= \int_0^1 \tan^{-1}x \ dx - \int_0^1 \tan^{-1}(x-1)dx \qquad \text{...(1)}$$

Observe how simple the integral I has now become. A further simplification is possible by the application of property - 9 on the second integral in (1):

$$\int_0^1 \tan^{-1}(x-1) = \int_0^1 \tan^{-1}\left((1-x)-1\right)dx$$

$$= -\int_0^1 \tan^{-1}x \ dx$$

Thus, $$I = 2\int_0^1 \tan^{-1}x \ dx$$

We can now proceed to evaluate I using integration by parts:

$$I = 2\left\{x\tan^{-1}x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}dx\right\}$$

$$= 2\left\{\frac{\pi}{4} - \frac{1}{2}\int_1^2 \frac{dt}{t}\right\} \qquad \left(\begin{matrix}\text{We used the substitution} \\ 1 + x^2 = t\end{matrix}\right)$$

$$= \frac{\pi}{2} - (\ln t)\Big|_1^2$$

$$= \frac{\pi}{2} - \ln 2$$

◀

Example – 5

Evaluate $I = \int_0^{\pi} \frac{x\sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$

Solution: By property - 9, we use the substitution $x \to \pi - x$ in the function to be integrated:

$$I = \int_0^{\pi} \frac{(\pi-x)\sin(2\pi-2x)\sin\left(\frac{\pi}{2}\cos(\pi-x)\right)}{2(\pi-x)-\pi}dx$$

$$= \int_0^{\pi} \frac{(\pi-x)\cdot(-\sin 2x)\cdot\left(-\sin\left(\frac{\pi}{2}\cos x\right)\right)}{\pi-2x}dx$$

$$= \int_0^{\pi} \frac{(x-\pi)\sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$$

Adding the original and the modified forms of I, we obtain:

$$2I = \int_0^{\pi} \frac{(2x-\pi)\sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$$

$$= \int_0^{\pi} \sin 2x \sin\left(\frac{\pi}{2}\cos x\right)dx$$

We use the substitution $\frac{\pi}{2}\cos x = t$ in this simplified integral. The limits change from $(0 \text{ to } \pi)$ to $\left(\frac{\pi}{2} \text{ to} \frac{-\pi}{2}\right)$:

$$2I = 2\int_0^{\pi} \sin x \cos x \sin\left(\frac{\pi}{2}\cos x\right)dx$$

$$= \frac{-8}{\pi^2}\int_{\pi/2}^{-\pi/2} t\sin t \, dt$$

$$= \frac{8}{\pi^2}\int_{-\pi/2}^{\pi/2} t\sin t \, dt$$

$$= \frac{8}{\pi^2}\left\{-t\cos t\Big|_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2}\cos t \, dt\right\}$$

$$= \frac{8}{\pi^2}\{0+2\}$$

$$= \frac{16}{\pi^2}$$

$$\Rightarrow \qquad I = \frac{8}{\pi^2}$$

Example – 6

Evaluate $I = \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} dx$

Solution: The limits of integration for I are symmetric about $x = 0$; this suggests that we should look for even/odd portions present, if any, in the function to be integrated:

$$I = 2\overbrace{\int_{\pi}^{\pi} \frac{x\sin x}{1+\cos^2 x} dx}^{\text{even}} + 2\overbrace{\int_{-\pi}^{\pi} \frac{x}{1+\cos^2 x} dx}^{\text{odd}}$$

$$= 4\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx + 0 \qquad \text{...(1)}$$

$$= 4\int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)} dx$$

$$= 4\int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} dx \qquad \text{...(2)}$$

Adding the expressions for I in (1)and (2), we obtain:

$$2I = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

Now use the substitution $\cos x = t$; $\sin x\ dx$ becomes $-dt$ and the limits of integration change from (0 to π) to (1 to –1):

$$I = 2\pi\int_1^{-1} \frac{-dt}{1+t^2}$$

$$= 2\pi(\tan^{-1} t)\Big|_{-1}^{1}$$

$$= \pi^2$$

◀

Example – 7

Prove that $I = \int_0^{2\pi} \frac{x\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx = \pi^2$.

Solution: Applying property -9 on I and then adding the original and modified forms of I, we obtain:

$$2I = 2\pi\int_0^{2\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$$

The function to be integrated is symmetric about π (verify this by the substitution $x \to 2\pi - x$). Therefore:

$$2I = 4\pi \int_0^{\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$$

The function to be integrated is still symmetric, about $x = \frac{\pi}{2}$; verify this by the substitution $x \to \pi - x$. Therefore:

$$2I = 8\pi \int_0^{\pi/2} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx \qquad ...(1)$$

Now we apply property -9 on the integral above, i.e., we use $x \to \frac{\pi}{2} - x$. Thus,

$$2I = 8\pi \int_0^{\pi/2} \frac{\cos^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$4I = 8\pi \int_0^{\pi/2} \frac{\sin^{2n} x + \cos^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$$

$$= 8\pi \int_0^{\pi/2} dx$$

$$= 4\pi^2$$

$$\Rightarrow \qquad I = \pi^2$$

◀

Example – 8

Find a lower bound and an upper bound for the integral

$$I = \int_0^1 \frac{1}{\sqrt{1+x^6}} dx$$

Solution: By this question, we mean that we need to find two values m and M between which I lies, i.e.,

$$m < I < M.$$

Technically, any number less than I, say –(1 billion) is a valid lower bound for I and any number greater than I, say (1 billion), would be a valid upper bound. But how useful would it be to state the (trivial) fact that I lies between (– 1 billion) and (1 billion)? Not much.

We want 'tight' bounds, i.e., narrow ranges in which I could lie. Thus, $M - m$ should be as small as possible so that we have an accurate idea about the approximate value of I.

Lets obtain an upper bound first:

We have,

$$1+x^6>1 \quad \text{for} \quad x\in(0,1)$$

$$\Rightarrow \quad \sqrt{1+x^6}>1 \quad \text{for} \quad x\in(0,1)$$

$$\Rightarrow \quad \frac{1}{\sqrt{1+x^6}}<1 \quad \text{for} \quad x\in(0,1)$$

$$\Rightarrow \quad \int_0^1 \frac{1}{\sqrt{1+x^6}}dx<\int_0^1 1\cdot dx$$

$$\Rightarrow \quad I<1$$

Thus we now know that I is less than 1.

Lets obtain a 'good' lower bound now:

Since $x\in(0,1)$, $x^6<x$. Thus,

$$1+x^6<1+x<(1+x)^2 \quad \text{for} \quad x\in(0,1)$$

$$\Rightarrow \quad \sqrt{1+x^6}<1+x \quad \text{for} \quad x\in(0,1)$$

$$\Rightarrow \quad \int_0^1 \frac{1}{\sqrt{1+x^6}}dx>\int_0^1 \frac{1}{1+x}dx$$

$$\Rightarrow \quad I>\ln 2$$

Thus, we now have a fair idea about the approximate value of I:

$$\ln 2<I<1$$

Try to obtain tighter bounds for yourself. ◀

Example – 9

Evaluate $I_n=\int_0^{\pi/2}\sin^n x\,dx$

Solution: Let $f_n(x)=\int \sin^n x\,dx$

$$\Rightarrow \quad f_n(x)dx=\int \sin^n x\,dx$$

$$=\int \underset{\substack{\downarrow\\ \text{Ist}\\ \text{function}}}{\sin^{n-1}x}\cdot \underset{\substack{\downarrow\\ \text{IInd}\\ \text{function}}}{\sin x}\ dx$$

$$=-\sin^{n-1}x\cos x+(n-1)\int \sin^{n-2}x\cdot\cos^2 x\,dx$$

$$=-\sin^{n-1}x\cos x+(n-1)\int \sin^{n-2}x(1-\sin^2 x)dx$$

$$= -\sin^{n-1} x \cos x + (n-1) f_{n-2}(x) - (n-1) f_n(x)$$

$$\Rightarrow \qquad nf_n(x) = -\sin^{n-1} x \cos x + (n-1) f_{n-2}(x)$$

$$\Rightarrow \qquad f_n(x) = \frac{-\sin^{n-1} x \cos x}{n} + \left(\frac{n-1}{n}\right) f_{n-2}(x)$$

$$\Rightarrow \qquad I_n = f_n(x)\Big|_0^{\pi/2} = 0 + \left(\frac{n-1}{n}\right) I_{n-2}$$

Thus, our recursive relation is

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}$$

We now use this repeatedly:

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}$$

$$= \frac{(n-1)(n-3)}{n(n-2)} I_{n-4}$$

$$\vdots$$

$$= \begin{cases} \dfrac{(n-1)(n-3)\ldots\ldots\ldots\ldots.(2)}{n(n-2)\ldots\ldots\ldots\ldots\ldots\ldots.(3)} I_1 & \text{(if } n \text{ is odd)} \\ \text{or} & \\ \dfrac{(n-1)(n-3)\ldots\ldots\ldots\ldots.(1)}{n(n-2)\ldots\ldots\ldots\ldots\ldots\ldots.(2)} I_0 & \text{(if } n \text{ is even)} \end{cases} \qquad \ldots(1)$$

Now, $\quad I_1 = \int_0^{\pi/2} \sin x \, dx = 1$

and $\quad I_0 = \int_0^{\pi/2} dx = \frac{\pi}{2}$

Thus, I_n is now obtainable from (1).

Notice that $J_n = \int_0^{\pi/2} \cos^n x \, dx$ will be the same as I_n, by virtue of property – 9. ◀

Example – 10

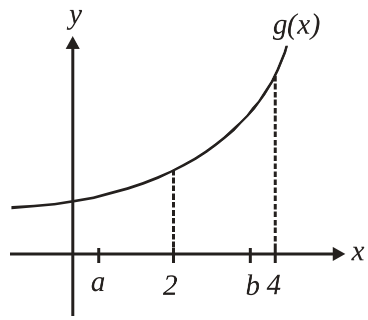
Suppose that $a + b = 4$ where $a \in (0, 2)$ and $g(x)$ is a differentiable function such that $g'(x) > 0 \;\; \forall \;\; x \in \mathbb{R}$.

Show that $\int_0^a g(x)dx + \int_0^b g(x)dx$ increases as $(b - a)$ increases:

Solution: Let us first intuitively try to justify the stated assertion.

Since $g'(x) > 0 \;\; \forall \;\; x \in \mathbb{R}$, $g(x)$ is an increasing function on $\mathbb{R}$.

Now, since $0 < a < 2$ and $a + b = 4$, a and b will lie symmetrically about the point $x = 2$. As a increases from 0 to 2, b decreases from 4 to 2. Assume an arbitrary (increasing) configuration for $g(x)$:



a and b lie symmetrically about the point x = 2; a∈(0, 2) and b∈(2, 4)

Fig - 20

Now we interpret the area given by $\int_0^a g(x)dx + \int_0^b g(x)dx$ from the figure below

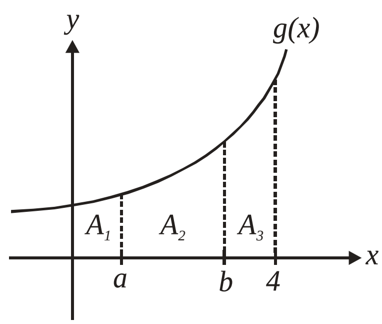


Fig - 21

We have
$$A = \int_0^a g(x)dx + \int_0^b g(x)dx$$
$$= A_1 + (A_1 + A_2)$$
$$= 2A_1 + A_2$$
$$= (A_1 + A_2 + A_3) - (A_3 - A_1) \qquad \left(\text{We introduced } A_3 \text{ into the expression}\right)$$
$$= \int_0^4 g(x)dx - (A_3 - A_1) \qquad \text{...(1)}$$

Now consider what will happen as $(b - a)$ increases. The area A_2 expands, while A_1 and A_3 will shrink (visualise this 'expansion' and 'shrinking' of areas from Fig.21). But the important point is that the shrink in A_3 and A_1 will not be the same. Since $g(x)$ is increasing, A_3 will shrink more than A_1 will. Thus, as $(b - a)$ increases, $(A_3 - A_1)$ will decrease.

From (1), therefore, A will increase!

The maximum A is obtained when $A_3 - A_1 = 0,$ i.e.,

When $a = 0$ and $b = 4$:

$$A_{\max} = \int_0^4 g(x)dx$$

Now we redo this example analytically. We treat a as a variable which can lie in (0, 2). Also, $b = 4 - a$.

Thus,

$$A = \int_0^a g(x)dx + \int_0^b g(x)\,dx$$

$$= \int_0^a g(x)dx + \int_0^{4-a} g(x)\,dx$$

$$\Rightarrow \quad \frac{dA}{da} = g(a) - g(4-a)$$

Since $g(x)$ is an increasing function and $a < 4 - a$, we have $g(a) < g(4-a)$ so that

$$\frac{dA}{da} < 0$$

Thus, A decreases with respect to a or equivalently, increases with respect to $b - a$. ◀

Example – 11

Evaluate these limits:

(a) $L_1 = \lim\limits_{x \to 0} \dfrac{\int_0^x (1 - \cos 2x)\,dx}{x \int_0^x \tan x\,dx}$ **(b)** $L_2 = \lim\limits_{x \to 0} \dfrac{\left(\int_0^x e^x dx \right)^2}{\int_0^x e^{x^2}\,dx}$

Solution: Observe that both these limits are of the from $\dfrac{0}{0}$, and therefore, we can use the L.H. rule. To differentiate the integrals, we can use the Leibnitz's differentiation rule.

(a)

$$L_1 = \lim_{x \to 0} \frac{1 - \cos 2x}{x \tan x + \int_0^x \tan x\,dx} \qquad \left(\text{still of the form } \frac{0}{0} \right)$$

$$= \lim_{x\to 0} \frac{2\sin 2x}{x\sec^2 x + \tan x + \tan x}$$

$$= \lim_{x\to 0} \frac{4\sin x\cos x}{x\sec^2 x + 2\tan x}$$

$$= \lim_{x\to 0} \frac{4\sin x\cos^3 x}{x + 2\sin x\cos x}$$

$$= \lim_{x\to 0} \frac{4\cos^3 x}{\left(\frac{x}{\sin x}\right) + 2\cos x}$$

$$= \frac{4}{1+2}$$

$$= \frac{4}{3}.$$ ◀

(b)

$$L_2 = \lim_{x\to 0} \frac{\not{2}\left(\int_0^x e^x dx\right).e^x}{\not{2}x\cdot e^{x^4}} \qquad \left(\text{still of the form } \frac{0}{0}\right)$$

$$= \lim_{x\to 0}\left(\frac{e^x}{e^{x^4}}\right)\cdot \lim_{x\to 0}\left(\frac{\int_0^x e^x dx}{x}\right) \qquad \left\{\begin{array}{l}\text{The second limit is of} \\ \text{the form } \frac{0}{0}\end{array}\right\}$$

$$= 1\cdot \lim_{x\to 0}\frac{e^x}{1}$$

$$= 1$$ ◀

Example – 12

Find a function $g(x)$ continuous in $(0,\infty)$ such that $g(x) > 0 \;\; \forall \;\; x \in (0,\infty)$ and $g(0) = 0$, and

$$\int_0^x g^2(t)dt = \frac{2}{x}\left(\int_0^x g(t)dt\right)^2$$

Solution: Differentiating the given relation w.r.t. x, we obtain:

$$g^2(x) = \frac{4}{x}\left(\int_0^x g(t)dt\right)\cdot g(x) - \frac{2}{x^2}\left(\int_0^x g(t)dt\right)^2$$

Let $\int_0^x g(t)dt = y$. Thus,

$$g^2(x) = \frac{4y\,g(x)}{x} - \frac{2y^2}{x^2}$$

$$\Rightarrow \quad 2y^2 - 4xg(x)y + x^2g^2(x) = 0$$

$$\Rightarrow \quad y = \frac{4xg(x) \pm \sqrt{16x^2g^2(x) - 8x^2g^2(x)}}{4}$$

$$= \frac{\left(4 \pm 2\sqrt{2}\right)xg(x)}{4}$$

$$= \left(1 \pm \frac{1}{\sqrt{2}}\right)x\ g(x)$$

Thus,

$$\int_0^x g(t)dt = kx\,g(x) \qquad \left\{k = 1 \pm \frac{1}{\sqrt{2}}\right\}$$

Differentiating both sides, we obtain:

$$g(x) = k\left\{g(x) + xg'(x)\right\}$$

$$\Rightarrow \quad (1-k)g(x) = \frac{kx\,d\,g(x)}{dx}$$

$$\Rightarrow \quad \frac{dg(x)}{g(x)} = \left(\frac{1-k}{k}\right)\frac{dx}{x}$$

Integrating both sides, we obtain,

$$\ln g(x) = \left(\frac{1-k}{k}\right)\ln x + C$$

$$\Rightarrow \quad g(x) = C_0 x^{\frac{1-k}{k}} \qquad \left(\begin{array}{l}\text{Take } C \text{ as } \ln C_0 \text{ where} \\ C_0 \text{ is another constant}\end{array}\right)$$

Since $g(0) = 0,$ the power of x must be positive, i.e.,

$$k = 1 - \frac{1}{\sqrt{2}}$$

Thus,

$$g(x) = C_0 x^{\frac{1}{\sqrt{2}-1}} = C_0 x^{\sqrt{2}+1}$$

◀

Example – 13

Prove that if $k \in \mathbb{Z}^+$,

$$\frac{\sin 2kx}{\sin x} = 2\left[\cos x + \cos 3x + ... + \cos(2k-1)x\right]$$

Hence or otherwise, prove that

$$\int_0^{\pi/2} \sin 2kx \cot x dx = \pi/2$$

Solution: Let us first directly try to prove the second part using the technique of recursion.

Let $I_k = \int_0^{\pi/2} \sin 2kx \cot x\, dx$

$$\Rightarrow \quad I_1 = \int_0^{\pi/2} \sin 2x \cot x\, dx$$

$$= 2\int_0^{\pi/2} \cos^2 x\, dx$$

$$= 2\int_0^{\pi/2} \left[\frac{1+\cos 2x}{2}\right] dx$$

$$= \frac{\pi}{2}.$$

Also, $I_{k+1} - I_k = \int_0^{\pi/2} \left(\sin(2k+2)x - \sin 2kx\right) \cot x\, dx$

$$= 2\int_0^{\pi/2} \cos(2k+1)x \sin x \cdot \cot x\, dx$$

$$= \int_0^{\pi/2} \left(2\cos x \cdot \cos(2k+1)x\right) dx$$

$$= \int_0^{\pi/2} \left(\cos(2k+2)x + \cos 2kx\right) dx$$

$$= \left.\frac{\sin(2k+2)x}{2k+2}\right|_0^{\pi/2} - \left.\frac{\sin 2kx}{2k}\right|_0^{\pi/2}$$

$$= 0 - 0$$

$$= 0$$

Thus,

$$I_k = \frac{\pi}{2} \quad \text{for all } \ k \in \mathbb{Z}^+$$

Now we'll use the first result mentioned in the question to prove the second part.

The proof of the first result is simple:

$$2\sin x\left[\cos x+\cos 3x+..........+\cos(2k-1)x\right]$$

$$=\sin 2x+(\sin 4x-\sin 2x)+.....+(\sin 2kx-\sin(2k-2)x)$$

$$=\sin 2kx$$

Thus, the stated assertion is valid

Now,

$$I=\int_0^{\pi/2}\sin 2kx\cot x\,dx$$

$$=2\int_0^{\pi/2}\cos x\left[\cos x+\cos 3x+......\cos(2k-1)x\right]dx \qquad \begin{Bmatrix}\text{Using the}\\ \text{first result}\end{Bmatrix}$$

$$=\int_0^{\pi/2}\{1+\cos 2x+(\cos 4x-\cos 2x)+......+\cos 2kx-\cos(2k-2)x\}dx$$

$$=\int_0^{\pi/2}(1+\cos 2kx)dx$$

$$=\frac{\pi}{2} \qquad \{\cos 2kx \text{ integrates to } 0\}$$

◀

Example – 14

Suppose that $g(x)$ is an even function, and $f(x)=\int_0^x g(t)dt.$

Is $f(x)$ even or odd, or neither?

Solution: $$f(-x)=\int_0^{-x}g(t)dt$$

If we let $t=-y,$ the limits of integration change from (0 to –x) to (0 to x). Thus,

$$f(-x)=\int_0^x g(-y)(-dy)$$

$$=-\int_0^x g(y)dy \qquad (\because\ g \text{ is even})$$

$$=-f(x)$$

Thus, $f(x)$ is odd. ◀

Example – 15

Let $x > 0$. Evaluate $\int_0^x [t]dt$.

Solution: The function to be integrated is discontinuous at all integer points. Therefore, we integrate it piecewise:

$$\int_0^x [t]dt = \int_0^1 [t]dt + \int_1^2 [t]dt + \ldots.. \int_{[x]-1}^{[x]} [t]dt + \int_{[x]}^x [t]dt$$

$$= 0 \cdot \int_0^1 dt + 1 \cdot \int_1^2 dt + 2 \cdot \int_2^3 dt + \ldots.. + ([x]-1) \cdot \int_{[x]-1}^{[x]} dt + \int_{[x]}^x [t]dt$$

$$= 0 + 1 + 2 + \ldots\ldots.([x]-1) + (x-[x])([x]) \qquad \begin{pmatrix}\text{Verify that the last}\\ \text{term is correct}\end{pmatrix}$$

$$= \frac{[x]([x]-1)}{2} + [x](x-[x])$$

◀