Straight Lines

CONCEPT NOTES

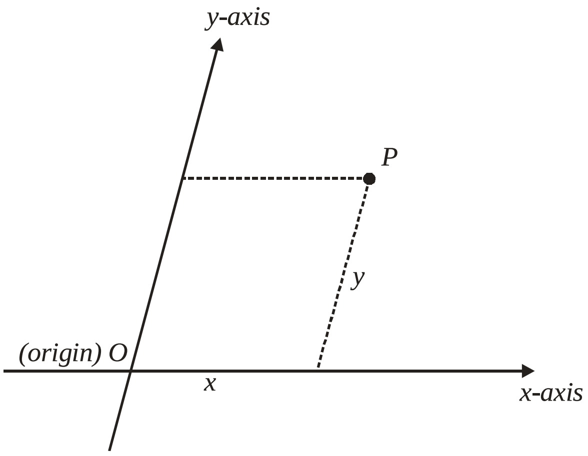
Straight Lines

Section - 1 INTRODUCTION

Co-ordinate geometry is a marriage of pure geometry and algebra, and is indispensable in all branches of science today. Many of you must be pretty familiar with the general outline of this subject.

We will restrict ourselves to 2-dimensional (plane) co-ordinate geometry in the following pages. Later on, we will also get the chance of studying 3-dimensional co-ordinate geometry.

The basic idea in co-ordinate geometry, as has been mentioned earlier, is to study the properties of geometrical figures such as straight lines, circles, parabolas etc through the use of numbers. The core concept is that on a 2-dimensional (Euclidean) plane, any point can be represented by a pair of real numbers, using two non-parallel straight lines. The point where these two non-parallel reference lines meet is termed the origin of the reference axis. By convention, one axis is called the x-axis and one the y-axis. Any point on the plane can now be determined in reference to this reference axes as described in the figure below:



The point P lies x units along the x-axis and y units along the y-axis, and therefore, P can be represented as (x, y). Note that x, y ε ℝ

x is called the abscissa of P while y is the ordinate of P. x and y together are called the co-ordinates of P.

Fig - 1

Conversely, given the co-ordinates x and y of a point P, we can easily determine its location by moving x units along the x-axis and then y-units parallel to the y-axis.

Notice that as long as the two axes are non-parallel, the entire plane is representable using these two axes as reference. These two axes in general can be at any non-zero angle to each other.

However, it is almost always the case (out of convenience) that the two axis are taken at right angles to each other. Such axes are called Rectangular Axes. We will always be using Rectangular Axes in our discussion from now onwards.

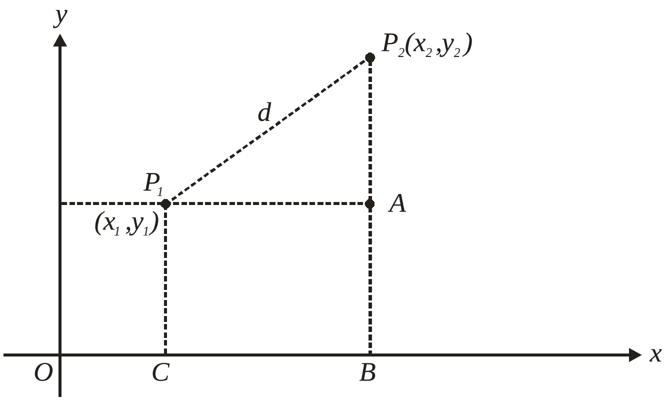
With this introduction, we start with the most elementary of geometric figures: line segments and lines (and other geometric figures obtainable from these elementary ones, like polygons).

Section - 2 STRAIGHT LINES

For a good command over co-ordinate geometry, a lot of results will be required to be memorised since they are encountered so often. For this purpose, each new theorem or result or property that we will encounter in the following pages is discussed in a separate article for ease of reference later.

Art 1 : Distance formula

One of the most basic expressions in co-ordinate geometry is that of the distance between two arbitrary points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. To obtain the required distance in terms of the co-ordinates of these points, the Pythagoras theorem is employed as described in the figure below:



Note that:
(i) $OC = x_1$; $OB = x_2$
$\Rightarrow BC = AP_1 = x_2 - x_1$

(ii) $CP_1 = y_1$; $BP_2 = y_2$
$\Rightarrow AP_2 = y_2 - y_1$

Fig - 2

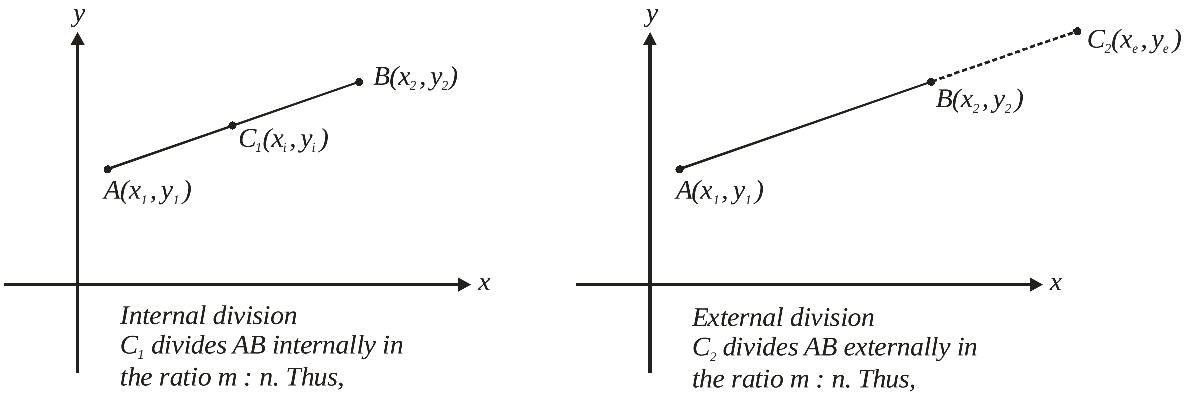
As explained in the figure, the distances AP_1 and AP_2 have been obtained. Thus, by the Pythagoras theorem, the distance d is $\sqrt{AP_1^2 + AP_2^2}$ or

$$\boxed{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

As a direct consequence of this formula, the distance of an arbitrary point $P(x, y)$ from the origin is $\sqrt{x^2 + y^2}$. As an elementary exercise, assume four points anywhere on the co-ordinate plane randomly, and use the distance formula to calculate the distance between each pair of points.

Art 2 : Section formula

Suppose that we are given two fixed points in the co-ordinate plane, say $A(x_1, y_1)$ and $B(x_2, y_2)$. We need to find the co-ordinates of the point C which divides the line segment AB in the ratio $m : n$. Observe that two such points will exist. Name them C_1 and C_2. One of them will divide the line segment *AB* ***internally*** in the ratio $m : n$ while the other will divide *AB* in the same ratio **externally**, as shown in the figure below:



Internal division
C_1 divides AB internally in the ratio m : n. Thus,

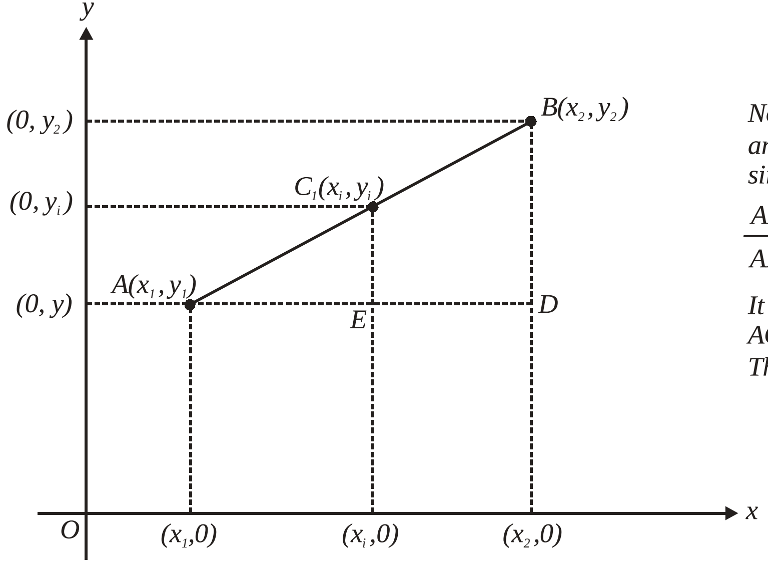
$$\frac{AC_1}{C_1B} = \frac{m}{n}$$

External division
C_2 divides AB externally in the ratio m : n. Thus,

$$\frac{AC_2}{C_2B} = \frac{m}{n}$$

Fig - 3

Let us find the co-ordinates of C_1 using the help of the more detailed figure of internal division below:



Note that $\Delta\ AEC_1$ and $\Delta\ ADB$ are similar. Thus,

$$\frac{AE}{AD} = \frac{EC_1}{DB} = \frac{AC_1}{AB}$$

It is given that $AC_1 : C_1B = m : n$. Thus,

$$\frac{AC_1}{AB} = \frac{m}{m+n}$$

Fig - 4

As described in the figure above,

$$\frac{AE}{AD} = \frac{EC_1}{DB} = \frac{m}{m+n}$$

$$\Rightarrow \qquad \frac{x_i - x_1}{x_2 - x_1} = \frac{y_i - y_1}{y_2 - y_1} = \frac{m}{m+n}$$

$$\Rightarrow \qquad x_i = \frac{mx_2 + nx_1}{m+n}, y_i = \frac{my_2 + ny_1}{m+n}$$

Thus, the co-ordinates of the point C_1 which divides *AB* internally in the ratio $m : n$ are

Internal Division m:n $$\boxed{\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}}$$

Using an analogous approach, we can obtain the co-ordinates of the point C_2 which divides *AB* externally in the ratio $m:n$

External Division m:n $$\boxed{\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}}$$

A particular case of internal division is finding the co-ordinates of the mid-point of *AB*. Since the mid-point of *AB* divides the segment *AB* in the ratio 1 : 1, the co-ordinates of the mid-point will be

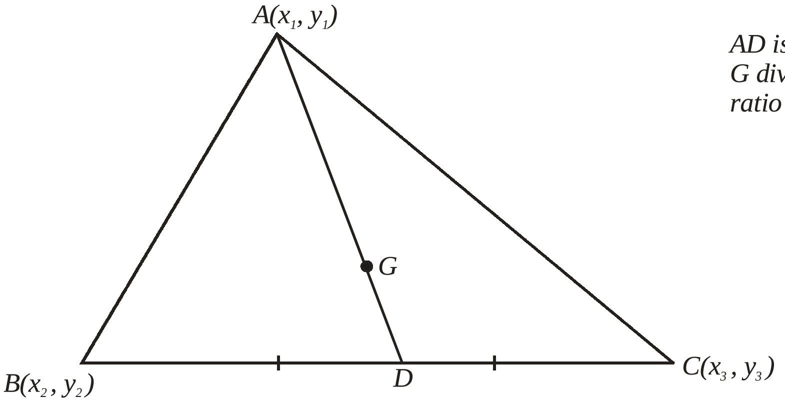
Mid – point of AB where $A \equiv (x_1, y_1)$ and $B \equiv (x_2, y_2)$ $$\boxed{\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}}$$

Let us put these rudimentary results to use.

Example – 1

Find the co-ordinates of the centroid of a triangle with the vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.

Solution: To determine the centroid, we will borrow a result from plane geometry that you might remember from high school: the centroid divides any median in the ratio 2 : 1.



AD is a median of ΔABC
G divides AD in the ratio 2 : 1, i.e,
AG : GD = 2 : 1

Fig - 5

The co-ordinates of *D*, the mid-point of *BC*, are $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$. Since $AG : GD = 2:1$, we have the co-ordinates of *G* by the section formula as

$$G \equiv \frac{2\left(\frac{x_2 + x_3}{2}\right) + 1.x_1}{2+1}, \frac{2\left(\frac{y_2 + y_3}{2}\right) + 1.y_1}{2+1}$$

$$\equiv \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}$$

◀

The expression for the centroid confirms the obvious fact that the co-ordinates of the centroid are ‘symmetric’ with respect to the co-ordinates of the three vertices of the triangle.

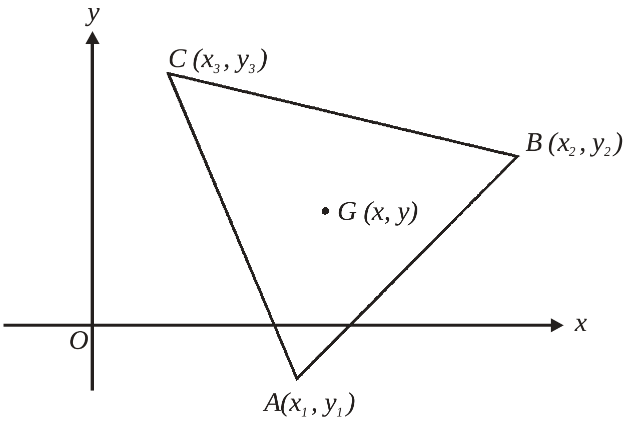
Example – 2

G is the centroid of triangle ABC. If O is any other point in the plane, prove that

$$OA^2 + OB^2 + OC^2 = GA^2 + GB^2 + GC^2 + 3GO^2.$$

Solution: There's no loss of generality in taking O as the origin of our reference axis since even if we are given O to be a non-origin point, we can always translate the axes (keeping the triangle ABC unchanged) so that its origin coincides with O. Note that this operation will have no effect on the lengths $OA, OB, OC, OG, GA, GB, GC$ etc. However, the expressions for distances will become much more simplified (In co-ordinate geometry, you will be required to follow such steps often, so that the expressions you are to deal with can be kept as simple as possible).

Now, we assume some co-ordinates for A, B and C as shown in the figure below:



As discussed earlier, the co-ordinates of G (x, y) are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Fig - 6

We have,

$$OA^2 + OB^2 + OC^2 \quad = \quad x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 + y_3^2 \qquad \text{... (1)}$$

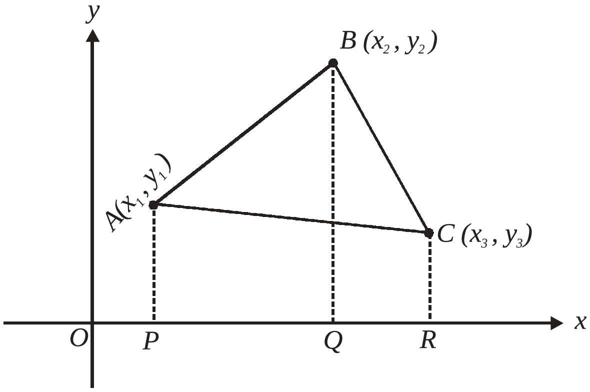
while

$$\begin{aligned} GA^2 + GB^2 + GC^2 + 3GO^2 &= (x - x_1)^2 + (y - y_1)^2 + (x - x_2)^2 + (y - y_2)^2 \\ &\quad + (x - x_3)^2 + (y - y_3)^2 + 3(x^2 + y^2) \\ &= x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2 \qquad \text{... (2)} \end{aligned}$$

Comparing (1) and (2), we see that the two expressions are indeed equal ◀

Art - 3 Area of a triangle

Suppose we are given three points in the co-ordinate plane : $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. We intend to find the area of ΔABC in terms of the given co-ordinates. How to evaluate this area is described in the figure below:



Note that
area (∆ABC) =
area (trapezium APQB)
+
area (trapezium BQRC)
−
area (trapezium APRC)

Fig - 7

Observe how the area of $\Delta\,ABC$ has been written in terms of the area of three trapeziums.

From plane geometry, the area of a trapezium is $\frac{1}{2}\times(\text{sum of bases})\times$ height. Thus,

$$\text{area}\left(\text{trap.}APQB\right)=\frac{1}{2}\times\left(AP+BQ\right)\times PQ$$

$$=\frac{1}{2}\left(y_1+y_2\right)\left(x_2-x_1\right) \qquad \text{... (1)}$$

Similarly,

$$\text{area}\left(\text{trap.}\,BQRC\right)=\frac{1}{2}\times\left(BQ+CR\right)\times QR$$

$$=\frac{1}{2}\left(y_2+y_3\right)\left(x_3-x_2\right) \qquad \text{... (2)}$$

$$\text{area}\left(\text{trap.}\,APRC\right)=\frac{1}{2}\times\left(AP+CR\right)\times PR$$

$$=\frac{1}{2}\left(y_1+y_3\right)\left(x_3-x_1\right) \qquad \text{... (3)}$$

From (1), (2) and (3), we have, upon simplification,

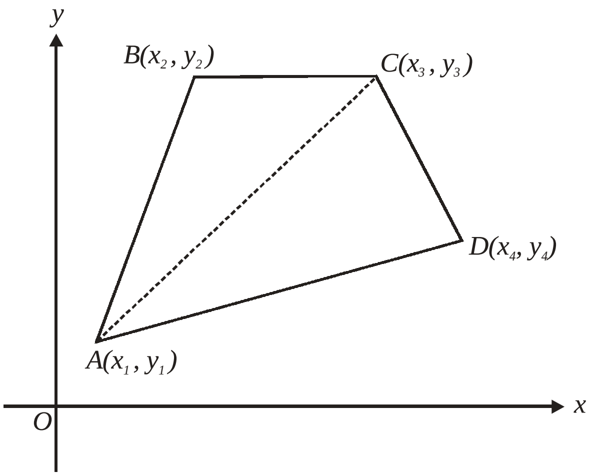
$$\text{area}\left(\Delta\,ABC\right)=\frac{1}{2}\left(x_2y_1-x_1y_2+x_3y_2-x_2y_3+x_1y_3-x_3y_1\right)$$

$$=\left|\frac{1}{2}\left\{x_1\left(y_2-y_3\right)+x_2\left(y_3-y_1\right)+x_3\left(y_1-y_2\right)\right\}\right|$$

We used the mod sign in the last expression because area is by definition positive.
We can express the area obtained in determinant form very concisely:

$$\Delta=\text{area}\left(\Delta\,ABC\right)=\left|\frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}\right|$$

As a consequence of this result, we can now easily find the area of an arbitrary quadrilateral *ABCD* as describe in the figure below:



area (quad ABCD)
= area (Δ ABC) + area (Δ ACD)

$$= \frac{1}{2}\left\{\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right| + \left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}\right|\right\}$$

In other words, we evaluate the areas of the two triangles separately and add them to get the area of the quadrilateral

Fig - 8

We can generalise this method easily to find the area of any polygon as a sum of the areas of the constituent triangles. ◀

Example – 3

Find the area of the triangle, the co-ordinates of whose vertices are $\left(ap, \frac{a}{p}\right), \left(aq, \frac{a}{q}\right)$ and $\left(ar, \frac{a}{r}\right)$.

Solution: Using the result obtained is Art - 3, we have,

$$\Delta = \left| \frac{1}{2} \begin{vmatrix} ap & \frac{a}{p} & 1 \\ aq & \frac{a}{q} & 1 \\ ar & \frac{a}{r} & 1 \end{vmatrix} \right|$$

$$= \frac{1}{2}\left| ap\left(\frac{a}{q} - \frac{a}{r}\right) + aq\left(\frac{a}{r} - \frac{a}{p}\right) + ar\left(\frac{a}{p} - \frac{a}{q}\right) \right|$$

$$= \frac{a^2}{2}\left| \frac{p(r-q)}{qr} + \frac{q(p-r)}{pr} + \frac{r(q-p)}{pq} \right|$$

$$= \frac{a^2\left|p^2(q-r) + q^2(r-p) + r^2(p-q)\right|}{2|pqr|}$$ ◀

Example – 4

Assume two fixed points in the co-ordinate plane: $A(a,0)$ and $B(-a,0)$. A variable point $C(x, y)$ moves in the plane in such a way that $CA + CB$ is a constant k. Use the distance formula to evaluate the condition that the co-ordinates of C must satisfy.

Solution: We have,

$$CA = \sqrt{(x-a)^2 + y^2}$$

and $$CB = \sqrt{(x+a)^2 + y^2}$$

From the constraint specified in the question, we have

$$CA + CB = k$$

$$\Rightarrow \quad CA^2 + CB^2 + 2CA \cdot CB = k^2$$

$$\Rightarrow \quad (x-a)^2 + y^2 + (x+a)^2 + y^2 + 2\sqrt{\left((x-a)^2 + y^2\right)\left((x+a)^2 + y^2\right)} = k^2$$

$$\Rightarrow \quad 2\sqrt{\left(x^2 - a^2\right)^2 + y^4 + 2y^2\left(x^2 + a^2\right)} = k^2 - 2\left(x^2 + y^2 + a^2\right)$$

$$\Rightarrow \quad 2\sqrt{x^4 + y^4 + a^4 + 2x^2y^2 - 2a^2x^2 + 2a^2y^2} = k^2 - 2\left(x^2 + y^2 + a^2\right)$$

Squaring both sides and cancelling out the common terms on both sides, we obtain

$$-8a^2x^2 = k^4 + 8a^2x^2 - 4k^2\left(x^2 + y^2 + a^2\right)$$

$$\Rightarrow \quad 4k^2x^2 - 16a^2x^2 + 4k^2y^2 = k^4 - 4k^2a^2$$

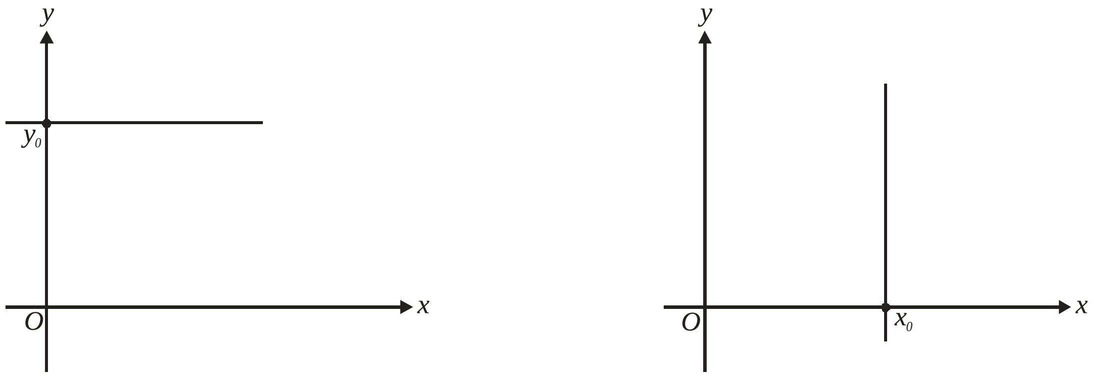
$$\Rightarrow \quad 4x^2\left(k^2 - 4a^2\right) + 4k^2y^2 = k^2\left(k^2 - 4a^2\right)$$

This is the relation that the co-ordinates of the variable point $C\,(x, y)$ must satisfy. All the pairs (x, y) which satisfy this equation, when plotted on the co-ordinate plane, will trace out the path on which the variable point C is constrained to move. In other words, this equation specifies the *locus* of the point C. ◀

Art - 4 Equation (s) representing a straight line

The last three articles dealt with the preliminaries of co-ordinate geometry and certain elementary formulae which find widespread use. With this article, we start the discussion of the geometry of straight lines in detail.

On the co-ordinate plane, the simplest case for a straight line would be one in which the line is parallel to one of the co-ordinate axes.



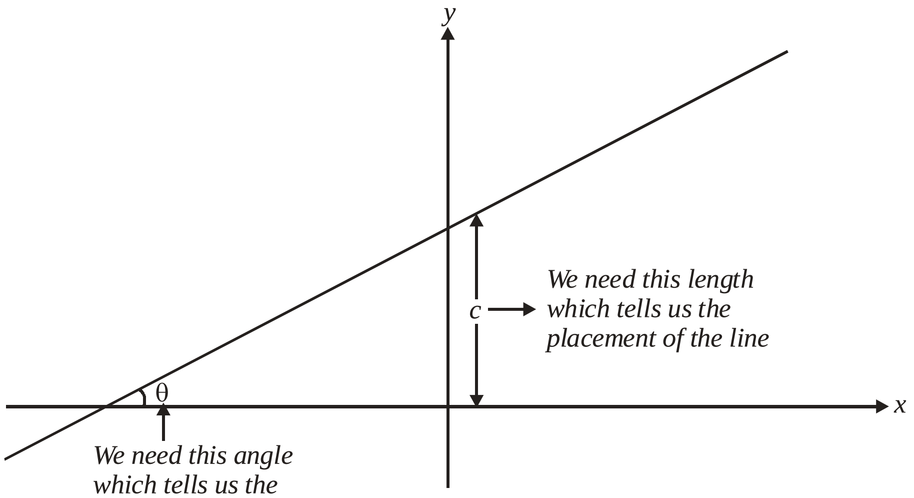
A line parallel to the x-axis.
Any point on this line has a constant y-co-ordinate equal to y_0.
Thus, we can say that the equation of this line is $y = y_0$
(There's no constraint on the x-co-ordinate of this line)

A line parallel to the y-axis.
Any point on this line has a constant x-co-ordinate equal to x_0.
Thus, we can say that the equation of this line is $x = x_0$
(There's no constraint on the y-co-ordinate of this line)

Fig - 9

As described in the figure above, the equation of such a line is $y = y_0$ or $x = x_0$ accordingly as the line is parallel to the *x*-axis or the *y*-axis respectively.

These are special cases of lines; we want to find the equation of any arbitrary line in general. Visualise any such line in your mind. To completely specify such a line, you would need two quantities: the inclination of the line (or its slope or the angle it makes with say, the *x*-axis) and the placement of the line (i.e. where the line passes through with reference to the axes: we can specify the placement of the line by specifying the point on the *y*-axis through which the line passes, or in other words, by specifying the *y*-intercept.)



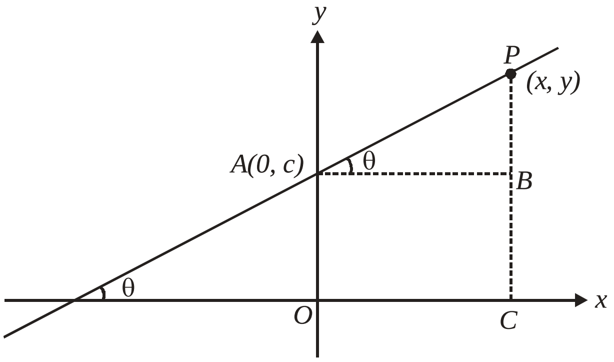
We need this length which tells us the placement of the line

We need this angle which tells us the inclination of the line

Fig - 10

It should be obvious to you that any line can be determined uniquely using these two parameters.

We now find out the equation of this straight line, assuming that we know θ and c. In other words, we intend to find out the relation that the co-ordinates (x, y) of any arbitrary point on the line must satisfy. The determination of this equation is straightforward:



We assume an arbitrary point P(x, y) on the line and try to relate x and y to the known quantities θ and c.
The relation we require is obtainable from the fact that in Δ APB,

$$\tan\theta = \frac{PB}{AB}$$

Fig - 11

As described in the figure above, we have in $\Delta\, APB$,

$$\tan\theta = \frac{PB}{AB}$$

tan θ is a measure of the inclination of the line (its steepness). tan θ is therefore termed the slope of the line and is denoted by *m*. Thus, $m = \tan\theta$. Also, notice that $PB = (y - c)$ and $AB = x$. Therefore,

$$m = \frac{y - c}{x}$$

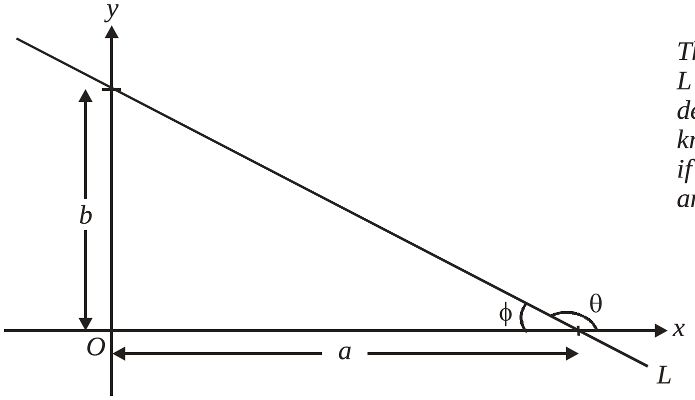
$$\Rightarrow \quad \boxed{y = \underset{\text{Slope}}{\underset{\nearrow}{m}}x + \underset{y-\text{intercept}}{\underset{\nearrow}{c}}} \qquad : \ \textbf{Slope - intercept form}$$

This is the general equation of a straight line involving its slope and its *y*-intercept. This form of the equation of the line is therefore termed the **slope-intercept** form.

Notice that if the line passes through the origin, its equation would reduce to $y = mx$.

As you might have guessed by now, this is not the only form to represent a straight line. This form uses the slope and the intercept of the line.

Lets discuss another form. Notice that to uniquely determine any straight line, we either need the slope of the line and a point through which this line passes, or we need at least two points through which that line passes. Thus for example, a line can also be uniquely determined if we are given the two points where this line intersects the *x*-axis and the *y*-axis.



The straight line L can be uniquely determined if we know a and b, i.e if we know the x-intercept and the y - intercept

Fig - 12

Notice that $\tan\phi = \frac{b}{a}$ so that the slope of the line is $m = \tan\theta = \tan(\pi - \phi) = -\tan\phi = -\frac{b}{a}$. Also, the y-intercept is b. Thus, using the slope intercept form obtained earlier, the equation of the line L is

$$y = -\frac{b}{a}x + b$$

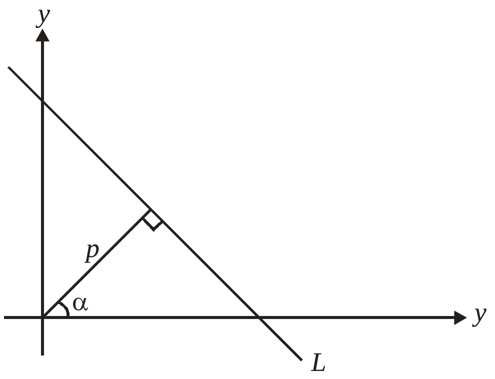
$$\Rightarrow \quad bx + ay = ab$$

$$\Rightarrow \quad \boxed{\frac{\underset{\nwarrow}{x}}{a} + \frac{y}{b} = 1} \qquad : \qquad \textbf{Intercept form}$$

$$x - \text{intercept} \qquad y - \text{intercept}$$

Thus, if we know the x and y intercepts, we can directly use this form to write the equation of the line.

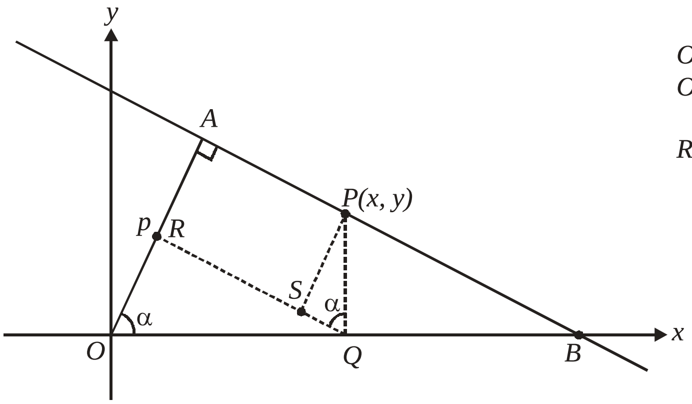
Lets consider a third form to represent a line. From the figure below, observe carefully that to uniquely determine a line, we could also specify the length of the perpendicular dropped from the origin to that line and the orientation (inclination) of that perpendicular:



The straight line L can be uniquely determined if we know p and α

Fig - 13

To determine the equation of this line, assume any point P on the line with the co-ordinate (x, y). Observe the geometry described in the figure below carefully:



Observe that:

OR = OQ cos α = x cos α

and

RA = SP = PQ sin α

= y sin α

Fig - 14

From the figure, note that

$$OR + RA = OA = p$$

$$\Rightarrow \quad \boxed{x\cos\underset{\substack{\nearrow \\ \textit{inclination of} \\ \textit{perpendicular}}}{\alpha} + y\sin\alpha = \underset{\substack{\nwarrow \\ \textit{length of} \\ \textit{perpendicular}}}{p}} \qquad : \quad \textbf{Normal form}$$

Thus, we now know of three forms in which the equation of an arbitrary straight line can be written.

From those three forms, you might be able to deduce that the most general form for the equation of an arbitrary straight line is $Ax + By + C = 0$. Let us try to prove this assertion, that is, let us try to show that $Ax + By + C = 0$ represents the equation of a straight line.

For this purpose, it will suffice to show that if we take any three arbitrary points $(x_1,\ y_1)$, $(x_2,\ y_2)$ and $(x_3,\ y_3)$ on the curve $Ax + By + C = 0,$ these three points will turn out to be collinear. Equivalently, the area of the triangle with the vertices as these three points will turn out to be zero.

Since all the three points satisfy the equation $Ax + By + C = 0,$ we have

$$Ax_1 + By_1 + C = 0$$

$$Ax_2 + By_2 + C = 0$$

$$Ax_3 + By_3 + C = 0$$

We can eliminate *A*, *B* and *C* from these three equations simultaneously to obtain a relation involving only the co-ordinates of the three points. A basic knowledge of elimination in determinant form will tell you that the relation we'll get after elimination is

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

which means that the area of the triangle formed by these three points as vertices is zero! Hence, the assertion is true.

With this discussion in mind, you should be able to write the equation for any arbitrary straight line. We will encounter the use of all these forms in the coming examples.

Before concluding this article, do this as a simple exercise based on the discussion we've already done:

(a) Show that the equation of the straight line with slope *m* and passing through the fixed point (x_1, y_1) is $y - y_1 = m(x - x_1)$

(b) Show that the equation of the straight line passing through the two fixed points $(x_1,\ y_1)$ and $(x_2,\ y_2)$ is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

The following table summarizes the various forms of the straight line that we've encountered.

	Known parameters about the line	**Equation**	**Name of this form**
1.	Slope m y-intercept c	$y = mx + c$	Slope-intercept form
2.	x -intercept a y -intercept b	$\frac{x}{a}+\frac{y}{b}=1$	Intercept form
3.	Length of perpendicular from origin to the line : p Inclination of perpendicular : α	$x\cos\alpha + y\sin\alpha = p$	Normal form
4.	Slope : m Any point through which the line passes : (x_1, y_1)	$y - y_1 = m(x - x_1)$	Point-slope form
5.	Any two points through : (x_1, y_1) which the line passes : (x_2, y_2)	$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$	Two point form
Most general form		: $Ax + By + C = 0$	where $A, B, C \in \mathbb{R}$ and at least one of A, B is non-zero

Note that each of the five specific forms mentioned in the table above can be converted easily to the most general form of the equation of a line. You are urged to do this as an exercise.

Also, the five forms are inter convertible among themselves in most cases too. For example, $y = mx + c$ can be written in intercept form as $\frac{x}{-(c/m)}+\frac{y}{c}=1$ so that the x-intercept of this line is $a = -\frac{c}{m}$ and the y-intercept is $b = c$. You are urged to try out all the (possible) conversions from one form to another.

You should now be able to understand that to determine a straight line uniquely, we must have two quantities given. Thus, two points could uniquely fix a line, or a point on the line and its slope could do so too, and so on. Notice that the general equation of the line also in fact contains only two arbitrary constants:

$$Ax + By + C = 0$$

$$\Rightarrow \quad \left(\frac{A}{C}\right)x + \left(\frac{B}{C}\right)y + 1 = 0$$

$$\Rightarrow \quad Px + Qy + 1 = 0 \qquad \left\{\begin{array}{l}\text{contains only two}\\ \text{arbitrary constants}\end{array}\right\}$$

Art 5 Point of intersection ; Angle of intersection

We are given two lines L_1 and L_2, and we are required to find the point at which they intersect (if they are non-parallel) and the angle at which they are inclined to one another, i.e., the angle of intersection. Evaluating the point of intersection is a simple matter of solving two simultaneous linear equations. Let the equations of the two lines be $\mathrm{L}_1 \equiv a_1x+b_1y+c_1=0$ and $L_2 \equiv a_2x+b_2y+c_2=0$ (written in the most general form). Now, let the point of intersection be (x_1, y_1). Thus,

$$a_1x_1+b_1y_1+c_1=0$$
$$a_2x_1+b_2y_1+c_2=0$$

This system can be solved to get

$$\frac{x_1}{b_1c_2-b_2c_1}=\frac{y_1}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}$$

From this relation we obtain the point of intersection $\left(x_1, y_1\right)$ as

$$\boxed{\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1},\frac{c_1a_2-c_2c_1}{a_1b_2-a_2b_1}\right)} \quad : \quad \textbf{Point of intersection}$$

To obtain the angle of intersection between these two lines, consider the figure below:

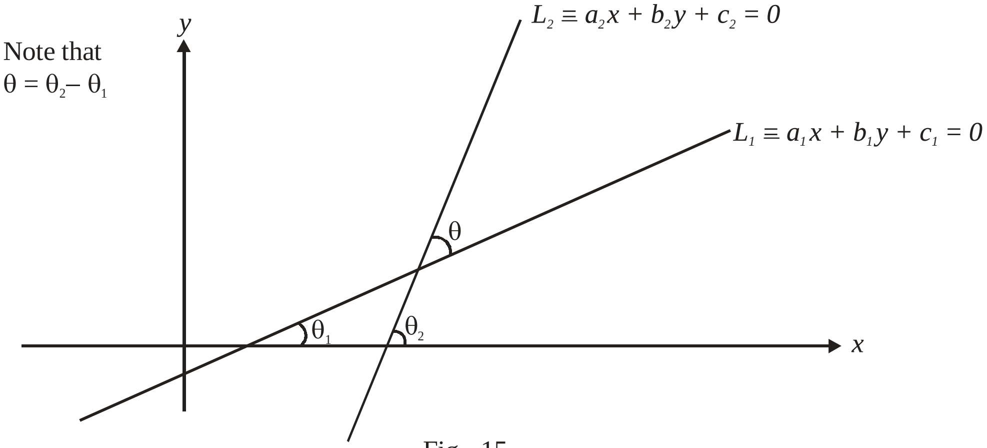


Fig - 15

The equation of the two lines in slope-intercept form are

$$y_1=\left(-\frac{a_1}{b_1}\right)x+\left(\frac{c_1}{b_1}\right)=m_1x+\left(\frac{c_1}{b_1}\right) \text{ where } m_1=-\frac{a_1}{b_1}$$

and $$y_2=\left(-\frac{a_2}{b_2}\right)x+\left(\frac{c_2}{b_2}\right)=m_2x+\left(\frac{c_2}{b_2}\right) \text{ where } m_2=-\frac{a_2}{b_2}$$

Note in Fig - 15 that $\theta=\theta_2-\theta_1$ and thus

$$\begin{aligned}\tan\theta &= \tan\left(\theta_2-\theta_1\right)\\ &= \frac{\tan\theta_2-\tan\theta_1}{1+\tan\theta_1\tan\theta_2}\\ &= \frac{m_2-m_1}{1+m_1m_2} \qquad \text{... (1)}\end{aligned}$$

Conventionally, we would be interested only in the acute angle between the two lines and thus we have to have $\tan\theta$ as a positive quantity. So in (1) above, if the expression $\frac{m_2-m_1}{1+m_1m_2}$ turns out to be negative, this would be the tangent of the obtuse angle between the two lines; thus, to get the acute angle between the two lines, we use the magnitude of this expression.

Therefore, the acute angle θ between the two lines is

$$\boxed{\theta = \tan^{-1}\left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|} \quad : \quad \textbf{Acute angle between the two lines}$$

From this relation, we can easily deduce the conditions on m_1 and m_2 such that the two lines L_1 and L_2 are parallel or perpendicular.

If the lines are parallel, $\theta = 0$ so that $m_1 = m_2$ which is intuitively obvious since parallel lines must have the same slope. For the two lines to be perpendicular, $\theta = \frac{\pi}{2}$ so that $\cot\theta = 0;$ this can happen if $1 + m_1 m_2 = 0$ or $m_1 m_2 = -1.$ Thus,

$$\boxed{m_1 = m_2 : \text{for parallel lines}}$$

and

$$\boxed{m_1\, m_2 = -1 : \text{for perpendicular lines}}$$

If the lines L_1 and L_2 are given in the general form given in the general form $ax + by + c = 0,$ the slope of this line is $m = -\frac{a}{b}$ so that the condition for L_1 and L_2 to be parallel becomes $-\frac{a_1}{b_1} = -\frac{a_2}{b_2}$ or $a_1 b_2 = a_2 b_1$ and the condition for L_1 and L_2 to be perpendicular becomes $\frac{a_1 a_2}{b_1 b_2} = -1$ or $a_1 a_2 + b_1 b_2 = 0$.

For example, the line $L_1 \equiv x + y = 1$ is perpendicular to the line $L_2 \equiv x - y = 1$ because the slope of L_1 is –1 while the slope of L_2 is 1.

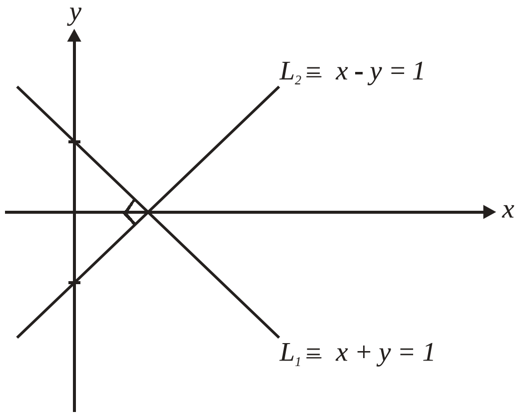


Fig - 16

As another example, the line $L_1 \equiv x - 2y + 1 = 0$ is parallel to the line $L_2 \equiv x - 2y - 3 = 0$ because the slope of both the lines is $m = \frac{1}{2}.$

Example – 5

Find the equation to the straight line which passes through (3, –2) and is inclined at 60º to the line $\sqrt{3}\,x+y=1.$

Solution: Observe carefully that there will be two such lines. Denote the two lines by L_1 and L_2

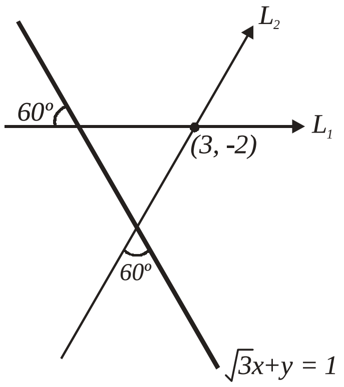


Fig - 17

Let the slope of the line(s) we require be *m*.

The slope of $\sqrt{3}\,x+y=1$ is $m_1=-\sqrt{3}$

Since we want the acute angle between the two lines to be 60º, we must have by Art - 5,

$$\tan 60^\circ=\left|\frac{m_1-m}{1+mm_1}\right|$$

$$\Rightarrow \qquad \sqrt{3}=\left|\frac{-\sqrt{3}-m}{1-\sqrt{3}\,m}\right|$$

$$\Rightarrow \qquad \frac{m+\sqrt{3}}{1-\sqrt{3}\,m}=\pm\sqrt{3}$$

$$\Rightarrow \qquad m+\sqrt{3}=\sqrt{3}-3m \text{ or } m+\sqrt{3}=3m-\sqrt{3}$$

$$\Rightarrow \qquad m=0 \text{ or } m=\sqrt{3}$$

Since we get two values of *m*, this confirms our earlier assertion that two such lines will exist. We now have the slope. We also know that the lines pass through (3, –2). We can therefore use the point-slope form to write down the required equations:

$$L_1\equiv y-(-2)=0(x-3);\; L_2\equiv y-(-2)=\sqrt{3}\,(x-3)$$

$$\Rightarrow \qquad L_1\equiv y+2=0 \text{ and } L_2\equiv y-\sqrt{3}\,x+2+3\sqrt{3}=0$$ ◀

Example – 6

Find the equation of the straight line which passes through the point $\left(a\cos^3\theta,\, a\sin^3\theta\right)$ and is perpendicular to the straight line $x\sec\theta+y\,\text{cosec}\theta=a$.

Solution: The slope of the given line is $m_1 = -\dfrac{\sec\theta}{\text{cosec}\theta} = -\tan\theta$

Therefore, the slope of the line we require will be given by m_2 where

$$m_2 = -\frac{1}{m_1}$$

$$\Rightarrow \qquad m_2 = \cot\theta$$

We now know the slope of the line and we are also given a fixed point through which the line passes. We can therefore use the point-slope form to determine its equation:

$$y - a\sin^3\theta = \cot\theta\left(x - a\cos^3\theta\right)$$

$$\Rightarrow \qquad x\cos\theta - y\sin\theta = a\left(-\sin^4\theta + \cos^4\theta\right)$$

$$= a\left(\cos^2\theta + \sin^2\theta\right)\left(\cos^2\theta - \sin^2\theta\right)$$

$$= a\cos 2\theta$$

Thus, the required equation is

$$x\cos\theta - y\sin\theta = a\cos 2\theta$$ ◀

Art 6 Half-planes

Any straight line divides the Euclidean plane into two half planes. In this article, we wish to determine the half-plane in which an arbitrary point lies with respect to a given line.

Let the equation of the given line be $ax + by + c = 0$. Consider two points (x_1, y_1) and (x_2, y_2) that lie in different half-planes with respect to this line:

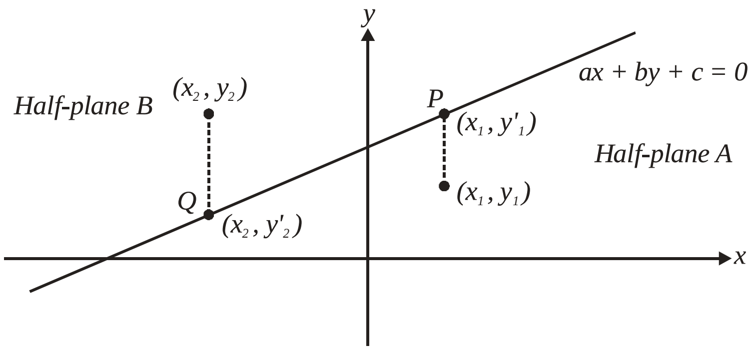


Fig - 18

The point (x_1, y_1) lies in the lower half-plane while (x_2, y_2) lies in the upper half plane. We require a condition on these co-ordinates which must be satisfied if the points lie in opposite half-planes. In Fig - 18, we have dropped verticle line segments from (x_1, y_1) and $(x_2 y_2)$ to the given line, intersecting it in *P* and *Q* respectively.

The co-ordinates of *P* and *Q* are $(x_1, y_1^{'})$ and $(x_2, y_2^{'})$ respectively where $y_1^{'} \neq y_1$ and $y_2^{'} \neq y_2$.

Since *P*, *Q* lie on the given line, their co-ordinates must satisfy the equation of the line. Thus,

$$ax_1 + by_1' + c = 0 \quad \Rightarrow \quad y_1' = -\frac{(ax_1 + c)}{b}$$

$$\text{and} \quad ax_2 + by_2' + c = 0 \quad \Rightarrow \quad y_2' = -\frac{(ax_2 + c)}{b}$$

Now, from Fig - 18 we have

$$y_1 < y_1' \quad \text{and} \quad y_2 > y_2'$$

$$\Rightarrow \quad y_1 < -\frac{(ax_1 + c)}{b} \quad \text{and} \quad y_2 > -\frac{(ax_2 + c)}{b}$$

$$\Rightarrow \quad \frac{ax_1 + by_1 + c}{b} < 0 \quad \text{and} \quad \frac{ax_2 + by_2 + c}{b} > 0$$

$\Rightarrow$ $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of opposite signs.

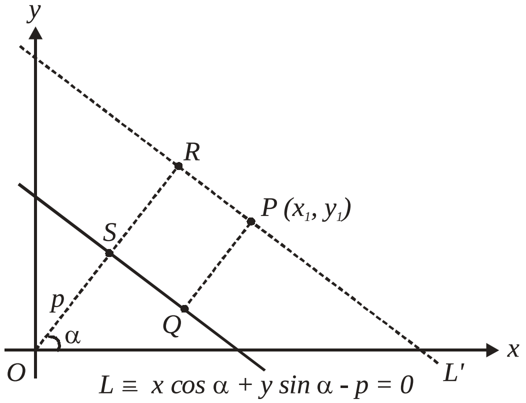
This is the required condition. Translated into words, it says that for two points lying in opposite half-planes, their co-ordinates when substituted respectively into the equation of the line must give expressions of opposite signs. (For two points in the same half-plane, the signs would be the same).

As a corollary, observe that a point (x_1, y_1) lies in the same half–plane or opposite half–plane in which the origin lies accordingly as $(ax_1 + by_1 + c)$ and c are of the same sign or opposite signs respectively.

Art 7 Length of perpendicular

Suppose that we are given the equation of a line L and we are required to find the length of the perpendicular dropped from an arbitrary point $P(x_1, y_1)$ on L.

Suppose that the equation of L is in normal form, i.e, $L \equiv x\cos\alpha + y\sin\alpha = p$.



We are required to find PQ. Note that
PQ = OR - OS
= OR - p
To determine OR, we draw a line L' parallel to L through P (x_1, y_1)
Let OR = p_1

Fig - 19

Based on the discussion in the figure above, the equation of the line L' is $x\cos\alpha + y\sin\alpha - p_1 = 0$. Since L_1 passes through P, the co-ordinates of P must satisfy the equation of L_1. Thus,

$$x_1\cos\alpha + y_1\sin\alpha - p_1 = 0$$

Thus, we get p_1 as $(x_1\cos\alpha + y_1\sin\alpha)$. The length of perpendicular PQ is now simply $|p_1 - p| = |x_1\cos\alpha + y_1\sin\alpha - p|$. (Modulus sign is used since PQ is a length so it must be positive).

$$\boxed{PQ = |x_1\cos\alpha + y_1\sin\alpha - p|} \; : \quad \textbf{Length of perpendicular}$$

Let us now assume the case where L is given in the general form, i.e. $L \equiv ax + by + c = 0$.

We can easily adjust the equation of L so that c is negative. We do this so that we can convert L into the normal form:

$$ax + by + c = 0 \qquad c < 0$$

$$\Rightarrow \quad ax + by = -c$$

$$\Rightarrow \quad \left(\frac{a}{\sqrt{a^2+b^2}}\right)x + \left(\frac{b}{\sqrt{a^2+b^2}}\right)y = \left(\frac{-c}{\sqrt{a^2+b^2}}\right)$$

$$\Rightarrow \quad x\cos\alpha + y\sin\alpha = p \qquad \ldots (1)$$

where $\cos\alpha = \dfrac{a}{\sqrt{a^2+b^2}}$, $\sin\alpha = \dfrac{b}{\sqrt{a^2+b^2}}$ and $p = \dfrac{-c}{\sqrt{a^2+b^2}}$

The equation in (1) is in the normal form; we can now use the result obtained in the preceding discussion to obtain the length of the perpendicular PQ:

$$PQ = |x_1\cos\alpha + y_1\sin\alpha - p| \qquad \left\{\begin{matrix}\text{Modulus sign is} \\ \text{used since } PQ \\ \text{must be +ve}\end{matrix}\right\}$$

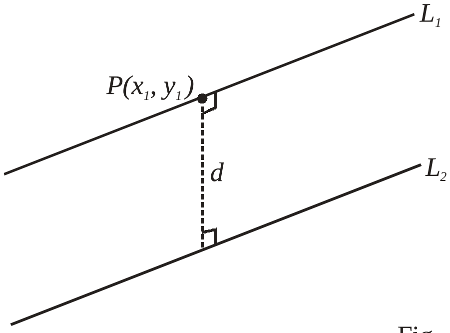
$$= \left|\frac{ax_1}{\sqrt{a^2+b^2}} + \frac{by_1}{\sqrt{a^2+b^2}} + \frac{c}{\sqrt{a^2+b^2}}\right|$$

$$\boxed{PQ = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}} \quad : \qquad \textbf{Length of perpendicular}$$

Example – 7

Find the distance between two parallel lines L_1 *and* L_2 given by $L_1 \equiv ax + by + c_1 = 0$ and $L_2 \equiv ax + by + c_2 = 0$

Solution:



Assume any point P on the line L_1; we are required to find d

Fig - 20

Since P lies on L_1, we have

$$ax_1 + by_1 + c_1 = 0$$

$$\Rightarrow \quad ax_1 + by_1 = -c_1$$

$$\Rightarrow \quad ax_1 + by_1 + c_2 = c_2 - c_1 \qquad \ldots (1)$$

By the previous article, the length of the perpendicular dropped from P upon the line L_2 is

$$d = \frac{|ax_1 + by_1 + c_2|}{\sqrt{a^2 + b^2}}$$

$$= \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}} \qquad \text{(By (1) above)}$$

This is the required distance between the two lines ◀

Example – 8

If p and p' be the perpendiculars from the origin upon the lines $L_1 \equiv x\sec\theta + y\,\text{cosec}\,\theta - a = 0$ and $L_2 \equiv x\cos\theta - y\sin\theta - a\cos 2\theta = 0,$ show that $4p^2 + p'^2 = a^2$

Solution: The length of the perpendicular dropped from (0, 0) to L_1 is by Art - 7

$$p = \frac{|a|}{\sqrt{\sec^2\theta + \text{cosec}^2\theta}}$$

$$= |a||\sin\theta\cos\theta| \qquad \text{... (1)}$$

Similarly, p', the length of the perpendicular from (0, 0) to L_2 is

$$p' = \frac{|a\cos 2\theta|}{\sqrt{\cos^2\theta + \sin^2\theta}}$$

$$= |a\cos 2\theta| \qquad \text{... (2)}$$

We now have from (1) and (2)

$$4p^2 + p^{'2} = 4a^2\sin^2\theta + a^2\cos^2 2\theta$$

$$= a^2\sin^2 2\theta + a^2\cos^2 2\theta$$

$$= a^2$$

◀

Art 8 Concurrency

Consider three different straight lines L_1, L_2 and L_3:

$$L_1 \equiv a_1x + b_1y + c_1 = 0 \qquad \text{... (1)}$$

$$L_2 \equiv a_2x + b_2y + c_2 = 0 \qquad \text{... (2)}$$

$$L_3 \equiv a_3x + b_3y + c_3 = 0 \qquad \text{... (3)}$$

We need to evaluate the constraint on the coefficients $a_i^{'}s,\ b_i^{'}s$ and $c_i^{'}s$ such that the three lines are concurrent.

Let us first determine the point P of intersection of L_1 and L_2. By Art - 5, it will be

$$P \equiv \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1},\ \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

Thus three lines will be concurrent if L_3 passes through P too, that is P satisfies the equation of L_3. Thus,

$$a_3\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}\right)+b_3\left(\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)+c_3=0$$

$$\Rightarrow \quad a_3\left(b_1c_2-b_2c_1\right)+b_3\left(c_1a_2-c_2a_1\right)+c_3\left(a_1b_2-a_2b_1\right)=0$$

$$\Rightarrow \quad a_1\left(b_2c_3-b_3c_2\right)+b_1\left(c_2a_3-c_3a_2\right)+c_1\left(a_2b_3-a_3b_2\right)=0$$

This relation can be written compactly in determinant form as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$$

This is the condition that must be satisfied for the three lines to be concurrent.

For example, consider the three lines $2x-3y+5=0,\ 3x+4y-7=0$ and $9x-5y+8=0$. These three lines are concurrent because the determinant of the coefficients is 0, i.e,

$$\begin{vmatrix} 2 & -3 & 5 \\ 3 & 4 & -7 \\ 9 & -5 & 8 \end{vmatrix}=0$$

Example – 9

Prove that the three lines L_1, L_2 and L_3 whose equations have been mentioned in the preceeding discussion, are concurrent if we can find three constants λ_1, λ_2 and λ_3 such that

$$\lambda_1L_1 + \lambda_2L_2+\lambda_3L_3=0$$

Solution: Assume that L_1 and L_2 intersect at the point P whose co-ordinates are (x_0, y_0) P should satisfy the equations of both L_1 and L_2.

$$L_1\left(\text{at } P\right)\equiv a_1x_0+b_1y_0+c_1=0 \qquad \text{... (1)}$$

$$L_2\left(\text{at } P\right)\equiv a_2x_0+b_2y_0+c_2=0 \qquad \text{... (2)}$$

Now assume that we can find three non-zero constants λ_1, λ_2 and λ_3 such that $\lambda_1L_1+\lambda_2L_2+\lambda_3L_3=0$. We will prove that due to this condition, L_3 will definitely have to pass through P:

$$\lambda_1L_1+\lambda_2L_2+\lambda_3L_3=0$$

$$\Rightarrow \quad L_3=\left(-\frac{\lambda_1}{\lambda_3}\right)L_1+\left(-\frac{\lambda_2}{\lambda_3}\right)L_2$$

If we evaluate the value of L_3 at P, we get

$$L_3\left(\text{at } P\right)=\left(-\frac{\lambda_1}{\lambda_3}\right)\times L_1\left(\text{at } P\right)+\left(-\frac{\lambda_2}{\lambda_3}\right)\times L_2\left(\text{at } P\right)$$

$$=\left(-\frac{\lambda_1}{\lambda_3}\right)\times 0+\left(-\frac{\lambda_2}{\lambda_3}\right)\times 0 \qquad \left\{\begin{matrix}\text{By (1)}\\ \text{and (2)}\end{matrix}\right\}$$

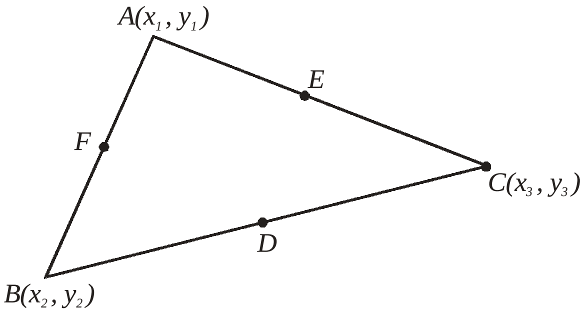
$$=0$$

Since the value of L is 0 at P the line L must pass through P. Thus, L L and L are concurrent ◀

Example – 10

Show that the medians of a triangle are concurrent.

Solution: Let the triangle have the vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ as in the figure below:



Let D, E, F be the mid-points of BC, CA and AB respectively. The co-ordinates of any mid point can easily be evaluated by the section formula. For eg, the co-ordinates of D are

$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$$

Fig - 21

From the two-point form of the equation of a line, we can write down the equations of *AD*, *BE* and *CF*.

The equation L_1 of the median *AD* is:

$$L_1 \equiv \frac{y-y_1}{x-x_1} = \frac{y_1 - \dfrac{y_2+y_3}{2}}{x_1 - \dfrac{x_2+x_3}{2}}$$

$$\Rightarrow \qquad L_1 \equiv \left(2y_1 - (y_2+y_3)\right)x - \left(2x_1 - (x_2+x_3)\right)y = x_1\left(2y_1 - (y_2+y_3)\right) - y_1\left(2x_1 - (x_2+x_3)\right)$$

By symmetry, we can write down the corresponding equations L_2 and L_3 of the medians *BE* and *CF*.

Observe carefully that when we subsequently add the three equations L_1, L_2 and L_3, their left hand sides sum to 0. Thus, we have found three constants 1, 1 and 1 such that

$$1\cdot L_1 + 1\cdot L_2 + 1\cdot L_3 = 0$$

$\Rightarrow$ L_1, L_2 and L_3 are concurrent

$\Rightarrow$ The medians of any triangle are concurrent. ◀

Example – 11

Show that the equation of any line passing through the intersection point *P* of two given lines whose equations are L_1 and L_2, can be expressed as $L_1 + \lambda L_2 = 0$, where λ is a real parameter.

Solution: Let $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$

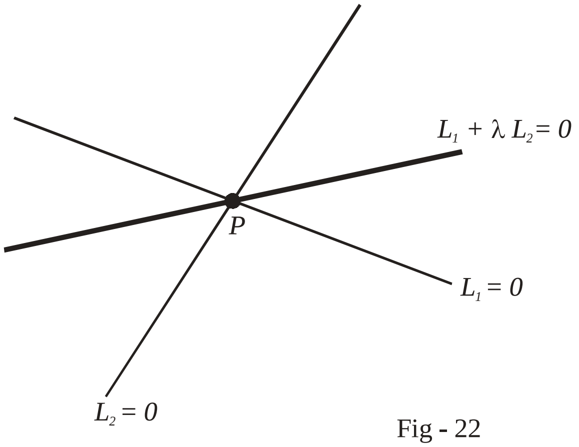
Consider the equation $L_1 + \lambda L_2 = 0$ ... (1)

$$\Rightarrow \qquad a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$$

$$\Rightarrow \qquad (a_1 + \lambda a_2)x + (b_1 + \lambda b_2)y + (c_1 + \lambda c_2) = 0$$

This is definitely the equation of a straight line because it is of the form $ax + by + c = 0$. Also, notice in addition that the intersection point *P* will satisfy this equation, because if we substitute the co-ordinates of the intersection point *P* in (1), both L_1 and L_2 vanish.

Thus, $L_1 + \lambda L_2 = 0$ is the equation of an arbitrary straight line that passes through the intersection point *P* of L_1 and L_2. (As we vary λ, the slope of this line will vary but it will **always** pass through *P*).



*The equation of **any** line passing through P can be written as*

$$L_1 + \lambda L_2 = 0$$

where λ is a real parameter

Fig - 22

This result is very beneficial in certain cases. We'll see such cases in some subsequent examples ◀

Example – 12

Find the equations to the straight lines passing through

(a) (3, 2) and the point of intersection of $2x + 3y = 1$ and $3x - 4y = 6$

(b) Origin and the point of intersection of $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$.

Solution: **(a)** The equations of the two given lines in standard form are :

$$L_1 \equiv 2x + 3y - 1 = 0$$

$$L_2 \equiv 3x - 4y - 6 = 0$$

Any line passing through the intersection point of L_1 and L_2 is

$$L_1 + \lambda L_2 = 0$$

$$\Rightarrow \quad (2x + 3y - 1) + \lambda(3x - 4y - 6) = 0$$

$$\Rightarrow \quad (2 + 3\lambda)x + (3 - 4\lambda)y - (1 + 6\lambda) = 0 \qquad \text{... (1)}$$

We want this line to pass through (3, 2). Therefore (3, 2) must satisfy the equation of this line, i.e.

$$(2 + 3\lambda)3 + (3 - 4\lambda)2 - (1 + 6\lambda) = 0$$

$$\Rightarrow \quad -5\lambda + 11 = 0$$

$$\Rightarrow \quad \lambda = \frac{11}{5}$$

We substitute $\lambda = \dfrac{11}{5}$ in (1) to get the required equation:

$$(2+2\cdot\frac{11}{5})x+(3-4\cdot\frac{11}{5})y-(1+6\cdot\frac{11}{5})=0$$

$$\Rightarrow \qquad 43x-29y-71=0$$

(b) We follow the same procedure as in part (a)

$$L_1 \ : \ bx+ay-ab=0$$

$$L_2 \ : \ ax+by-ab=0$$

The equation of any line passing through the intersection point of L_1 and L_2 is

$$L_1+\lambda L_2=0$$

$$\Rightarrow \qquad (b+\lambda a)x+(a+\lambda b)y-ab(1+\lambda)=0 \qquad \text{... (2)}$$

Since this line must pass through (0, 0), we substitute (0, 0) into (2) to get

$$ab(1+\lambda)=0$$

$$\Rightarrow \qquad \lambda=-1$$

We substitute $\lambda=-1$ into (2) to get the required equation :

$$(b-a)x+(a-b)y=0$$

$$\Rightarrow \qquad x-y=0$$

◀

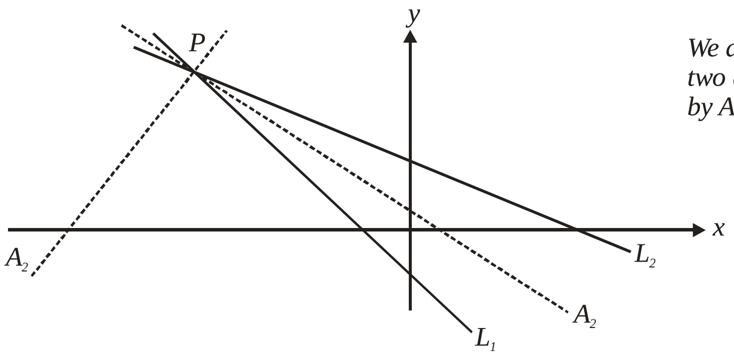
Art 9 $\boxed{\text{Angle Bisectors}}$

Consider two straight lines L_1 and L_2 with the equations

$$L_1 \ : \ a_1x+b_1y+c_1=0$$

$$L_2 \ : \ a_2x+b_2y+c_2=0$$

We intend to find the angle bisector formed at the intersection point P of L_1 and L_2. Note that there will be two such angle bisectors



We denote the two angle bisectors by A_1 and A_2

Fig - 23

To write down the equations of the two angle bisectors, we first modify the equations of L_1 and L_2 so that c_1 and c_2 are say, both negative in sign. This can always be done. Why this is done will soon become clear.

We first write down the equation of A_1, the angle bisector of the angle in which the origin lies.

By virtue of being an angle bisector, if any point $P(x', y')$ lies on A_1, the distance of P from L_1 and L_2 must be equal. Using the perpendicular distance formula of Art -7, we have

$$\Rightarrow \qquad \left|\frac{a_1x'+b_1y'+c_1}{\sqrt{a_1^2+b_1^2}}\right| = \left|\frac{a_2x'+b_2y'+c_2}{\sqrt{a_2^2+b_2^2}}\right|$$

$$\Rightarrow \qquad \frac{a_1x'+b_1y'+c_1}{\sqrt{a_1^2+b_1^2}} = \pm\frac{a_2x'+b_2y'+c_2}{\sqrt{a_2^2+b_2^2}} \qquad \text{...(1)}$$

Which sign should we select, "+" or "–", for the bisector of the angle containing the origin?

Since P and origin lie on the same side of L_1, $a_1x'+b_1y'+c_1$ and c_1 **must be** of the same sign by Art - 6. Similarly, $a_2x'+b_2y'+c_2$ and c_2 must be of the same sign. But since we have already arranged c_1 and c_2 to be of the same sign (both negative), we must have $\left(a_1x'+b_1y'+c_1\right)$ and $\left(a_2x'+b_2y'+c_2\right)$ also of the same sign.

Thus, it follows from (1) that to write the equation of the angle bisector of the angle containing the origin, we must select the "+" sign since $\left(a_1x'+b_1y'+c_1\right)$ and $\left(a_2x'+b_2y'+c_2\right)$ are of the same sign. The "–" sign gives the angle bisector of the angle not containing the origin, i.e., the equation of A_2.

To summarize, we first arrange the equations of L_1 and L_2 so that c_1 and c_2 are both of the same sign. Subsequently, using the property of any angle bisector, we obtain

$$\boxed{\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = +\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}}$$: **Angle bisector of angle contaning the origin**

and

$$\boxed{\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = -\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}}$$: **Angle bisector of angle not contaning the origin**

Example – 13

Find the angle bisector of the angle between the straight lines $L_1 : 3x-4y+7=0$ and $L_2 : 12x-5y-8=0$ which contains the origin.

Solution: Following the discussion of the preceeding article, we first modify the equations L_1 and L_2 so that the constant terms in both the equations are of the same sign (say both positive):

$$L_1 \ : \ 3x - 4y + 7 = 0$$

$$L_2 \ : \ -12x + 5y + 8 = 0$$

The angle bisector of the angle containing the origin is

$$\frac{(3x-4y+7)}{\sqrt{3^2+4^2}} = + \frac{(-12x+5y+8)}{\sqrt{12^2+5^2}}$$

$$\Rightarrow \qquad 99x - 77y + 51 = 0$$

Evaluating the other angle bisector is left to the reader as an exercise. ◀

Example – 14

Find the bisector of the angle between the lines $x + 2y - 11 = 0$ and $3x - 6y - 5 = 0$ which contains the point (1, –3).

Solution: Again we first arrange the equations of the two lines such that constant terms are positive

$$L_1 \ : \ -x - 2y + 11 = 0$$

$$L_2 \ : \ -3x + 6y + 5 = 0$$

Note that

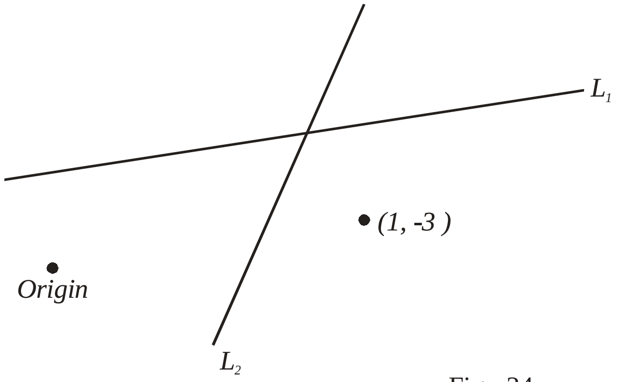
$$L_1(\text{at origin}) > 0 \text{ and } L_1(\text{at}(1,-3)) > 0$$

$\Rightarrow$ Origin and (1, –3) are on the same side of L_1.

$$L_2(\text{at origin}) > 0 \text{ and } L_2(\text{at}(1,-3)) < 0$$

$\Rightarrow$ Origin and (1, –3) are on the opposite sides of L_2.

This means that the point (1, –3) does not lie in the same region as the origin, since (1, –3) must be on the opposite side of the origin with respect to L_2. The example figure below will make this clear:



We see that (1, -3) lies in the angle not containing the origin

Fig - 24

Thus, it is clear that (1, –3) lies in the angle not containing the origin.

There is one point worth mentioning here. If suppose a point P (x, y) lies on the opposite side of the origin with respect to both L_1 *and* L_2*, it would lie in the vertically opposite angle to the angle in which the origin lies ; in such a case, the angle bisector of the angle containing P is the same as that of the one containing the origin.*

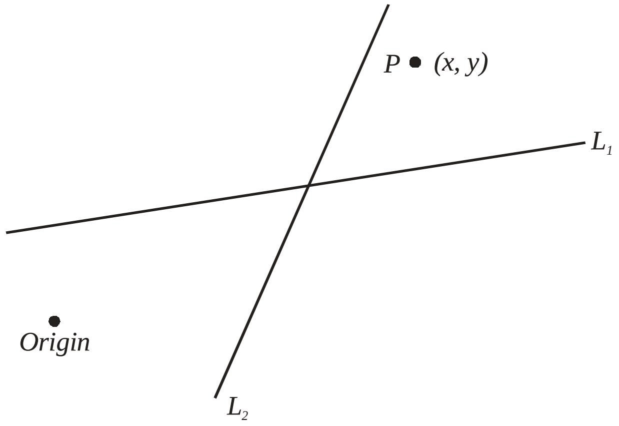


Fig - 25

To determine the angle bisector of the angle containing (1, –3), we simply determine the angle bisector of the angle not containing the origin, i.e.

$$\frac{-x-2y+11}{\sqrt{5}} = -\frac{-3x+6y+5}{3\sqrt{5}}$$

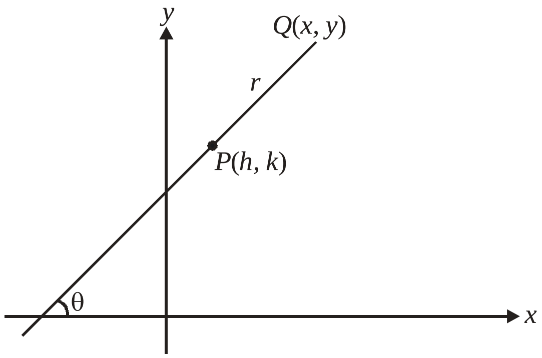
$$\Rightarrow \qquad 6\sqrt{5}x = 38\sqrt{5}$$

$$\Rightarrow \qquad x = \frac{19}{3}$$

Note that to determine the angle bisector of the angle containing the point *P* as in Fig.-25, we would have chosen the angle bisector of the angle **containing** the origin.

Art 10 Polar / Distance form of a line

Sometimes, it is very convenient to write the equation of a straight line in polar / distance form. Suppose we know that the line passes through the fixed point $P(h, k)$ and is at an inclination of θ:



Let $PQ = r$

Fig - 26

For any point $Q(x, y)$ at a distance r from P **along** this line, we can write the simple relation

$$\boxed{\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = r}$$

This is the required equation of the line. The point $Q(x, y)$, at a distance r from P, has the coordinates

$$Q(x, y) \equiv (h + r\cos\theta, k + r\sin\theta).$$

{Obviously, there will be another point, say $Q'(x, y)$, at a distance r from P along this line but on the opposite side of Q; thus $Q'(x, y)$ will have the coordinates $Q'(x, y) \equiv (h - r\cos\theta,\ k - r\sin\theta)$}

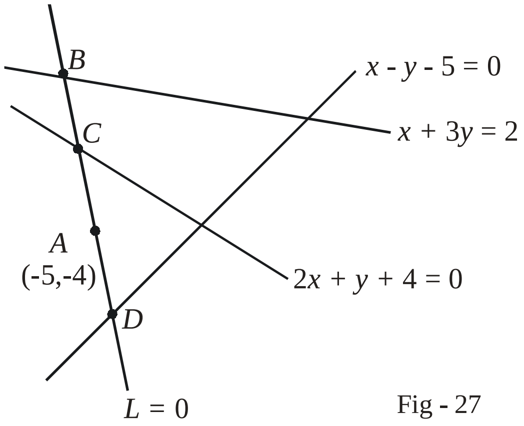
Example – 15

A line through $A(-5, -4)$ meets the lines $x + 3y = 2$, $2x + y + 4 = 0$ and $x - y - 5 = 0$ at the points B, C and D respectively. If

$$\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2,$$

find the equation of the line.

Solution:



We want to find the equation of the line *L*.
Assume
$AB = r_1$
$AC = r_2$
$AD = r_3$

Fig - 27

The figure above roughly sketches the situation described in the equation. Let *B*, *C* and *D* be at distances r_1, r_2 and r_3 from *A* along the line $L = 0$, whose equation we wish to determine. Assume the inclination of *L* to be θ. Thus, *B*, *C* and *D* have the coordinates (respectively):

$$B \equiv (-5 + r_1 \cos\theta, \quad -4 + r_1 \sin\theta)$$

$$C \equiv (-5 + r_2 \cos\theta, \quad -4 + r_2 \sin\theta)$$

$$D \equiv (-5 + r_3 \cos\theta, \quad -4 + r_3 \sin\theta)$$

Since these three points(respectively) satisfy the three given equations, we have :

Point *B* : $(-5 + r_1 \cos\theta) + 3(-4 + r_1 \sin\theta) + 2 = 0 \Rightarrow \qquad r_1 = \dfrac{15}{\cos\theta + 3\sin\theta}$

Point *C* : $2(-5 + r_2 \cos\theta) + (-4 + r_2 \sin\theta) + 4 = 0 \Rightarrow \qquad r_2 = \dfrac{10}{2\cos\theta + \sin\theta}$

Point *D* : $(-5 + r_3 \cos\theta) - (-4 + r_3 \sin\theta) - 5 = 0 \quad \Rightarrow \qquad r_3 = \dfrac{6}{\cos\theta - \sin\theta}$

It is given that

$$\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$$

i.e. $$\left(\frac{15}{r_1}\right)^2 + \left(\frac{10}{r_2}\right)^2 = \left(\frac{6}{r_3}\right)^2$$

$$\Rightarrow \qquad (\cos\theta + 3\sin\theta)^2 + (2\cos\theta + \sin\theta)^2 = (\cos\theta - \sin\theta)^2$$

$$\Rightarrow \qquad 4\cos^2\theta + 9\sin^2\theta + 12\sin\theta\cos\theta = 0$$

$$\Rightarrow \qquad (2\cos\theta + 3\sin\theta)^2 = 0$$

$$\Rightarrow \qquad \tan\theta = \frac{-2}{3}$$

$$\Rightarrow \qquad m = \frac{-2}{3}$$

Thus, we obtain the slope of L as $\frac{-2}{3}\cdot$ The equation of L can now be easily written :

$$L: y-(-4)=\frac{-2}{3}(x-(-5))$$

$$\Rightarrow \quad L: 2x+3y+22=0 \qquad ◀$$

TRY YOURSELF - I

1. A variable straight line drawn through the intersection of the lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$ meets the axes in *A* and *B*. Show that the locus of the mid-point of *AB* is $2xy(a+b)=ab(x+y)$

2. The line $bx+ay=ab$ cuts the axes in *A* and *B*. Another variable line cuts the axes in *C* and *D* such that $OA+OB=OC+OD,$ where *O* is the origin. Prove that the locus of the point of intersection of the lines *AD* and *BC* is the line $x+y=a+b.$

3. A point *P* moves so that the square of its distance from (3, –2) is equal to its distance from the line $5x-12y=13.$ Find the locus of *P*.

4. A line intersects the *x*-axis in *A*(7, 0) and the *y*-axis in *B*(0, –5). A variable line perpendicular to *AB* intersects the *x*-axis in *P* and the *y*-axis in *Q*. If *AQ* and *BP* intersect in *R*, find the locus of *R*.

5. If the sum of the distances of a point from two perpendicular lines in a plane is 1, prove that its locus is a square.

6. A vertex of an equilateral triangle is (2, 3) and the opposite side is $x+y=2.$ Find the equations of the other sides.

7. A ray of light along the line $x-2y-3=0$ is incident upon the mirror-line $3x-2y-5=0.$ Find the equation of the reflected ray.

8. If the vertices of a triangle have integral coordinates, show that it cannot be equilateral.

9. Show using coordinate geometry that the angle bisectors of the sides of a triangle are concurrent.

10. The sides of a triangle are $4x+3y+7=0,$ $5x+12y-27=0$ and $3x+4y+8=0.$ By explicitly evaluating the medians in this triangle, show that they are concurrent.

11. A rod *APB* of constant length meets the axes in *A* and *B*. If *AP* = *b* and *PB* = *a* and the rod slides between the axes, show that the locus of *P* is $b^2x^2+a^2y^2=a^2b^2$

12. If *p* is the length of the perpendicular from the origin to the line whose intercepts on the axes are *a* and *b*, show that $\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}.$

13. The lines $3x+4y-8=0$ and $5x+12y+3=0$ intersect in *A*. Find the equations of the lines passing through $P(3,4),$ which intersect the given lines at *B* and *C*, such that $AB=AC.$

14. The equal sides *AB* and *AC* of an isosceles triangle *ABC* are produced to the points *P* and *Q* such that $BP\cdot CQ=AB^2.$ Prove that the line *PQ* always passes through a fixed point.

15. One side of a square is inclined to the *x*-axis at an angle α and one of its extremities is at the origin; prove that the equations to its diagonals are

$$y(\cos\alpha-\sin\alpha)=x(\sin\alpha+\cos\alpha)$$

and $$y(\sin\alpha+\cos\alpha)+x(\cos\alpha-\sin\alpha)=a$$

where *a* is the length of the side of the sqaure.