Differential Equations

CONCEPT NOTES

Differential Equations

Section - 1 MOTIVATION

A differential equation can simply be said to be an equation involving derivatives of an unknown function. For example, consider the equation

$$\frac{dy}{dx} + xy = x^2$$

This is a differential equation since it involves the derivative of the function $y(x)$ which we may wish to determine. We must first understand why and how differential equations arise and why we need them at all. In general, we can say that a differential equation describes the behaviour of some continuously varying quantity.

Scenario - 1 : A freely falling body

A body is released at rest from a height h. How do we described the motion of this body ?

The height x of the body is a function of time. Since the acceleration of the body is g, we have

$$\frac{d^2x}{dt^2} = -g$$

This is the differential equation describing the motion of the body. Along with the initial condition $x(0) = h,$ it completely describes the motion of the body at all instants after the body starts falling.

Scenario - 2 : Radioactive disintegration

Experimental evidence shows that the rate of decay of any radioactive substance is proportional to the amount of the substance present, i.e.,

$$\frac{dm}{dt} = -\lambda m$$

where m is the mass of the radioactive substance and is a function of t. If we know $m(0)$, the initial mass, we can use this differential equation to determine the mass of the substance remaining at any later time instant.

Scenario - 3 **Population growth**

The growth of population (of say, a biological culture) in a closed environment is dependent on the birth and death rates. The birth rate will contribute to increasing the population while the death rate will contribute to its decrease. It has been found that for low populations, the birth rate is the dominant influence in population growth and the growth rate is linearly dependent on the current population. For high populations, there is a competition among the population for the limited resources available, and thus the death rate becomes dominant. Also, the death rate shows a quadratic dependence on the current population.

Thus, if $N(t)$ represents the population at time *t*, the differential equation describing the population variation is of the form

$$\frac{dN}{dt} = \lambda_1 N - \lambda_2 N^2$$

where λ_1 and λ_2 are constants.

Along with the initial population $N(0)$, this equation can tell us the population at any later time instant.

These three examples should be sufficient for you to realise why and how differential equations arise and why they are important.

In all the three equations mentioned above, there is only independent variable (the time *t* in all the three cases). Such equations are termed **ordinary differential equations**. We might have equations involving more than one independent variable:

$$\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = x^2$$

where the notation $\frac{\partial}{\partial x}$ stands for the partial derivative, i.e., the term $\frac{\partial f}{\partial x}$ would imply that we differentiate the function *f* with respect to the independent variable *x* as the variable (while treating the other independent variable *y* as a constant). A similar interpretation can be attached to $\frac{\partial}{\partial y}$.

Such equations are termed **partial differential equations** but we'll not be concerned with them in this chapter.

Consider the ordinary differential equation

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2 = c$$

The order of the highest derivative present in this equation is two; thus, we'll call it a second order differential equation (*DE*, for convenience).

The **order** of a DE is the order of the highest derivative that occurs in the equation

Again, consider the DE

$$\left(\frac{d^3y}{dx^3}\right)^2 + \frac{dy}{dx} = x^2y^2$$

The degree of the highest order derivative in this DE is two, so this is a DE of degree two (and order three).

The **degree** of a DE is the degree of the highest order derivative that occurs in the equation, **when all the derivatives in the equation are made of free of fractional powers**.

For example, the DE

$$\sqrt{\left(\frac{dy}{dx}\right)^2 - 1} + x\left(\frac{d^2y}{dx^2}\right)^2 = k$$

is not of degree two. When we make this equation free of fractional powers, by the following rearrangement,

$$\left(\frac{dy}{dx}\right)^2 - 1 = \left\{k - x\left(\frac{d^2y}{dx^2}\right)^2\right\}^2$$

we see that the degree of the highest order derivative will become four. Thus, this is a DE of degree four (and order two).

Finally, an n^{th} **linear DE** (degree one) is an equation of the form

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}\frac{dy}{dx} + a_ny = b$$

where the $a_i's$ and b are functions of x.

Solving an n^{th} order DE to evaluate the unknown function will essentially consist of doing n integrations on the DE. Each integration step will introduce an arbitrary constant. Thus, you can expect in general that the **solution of an n^{th} order DE will contain n independent arbitrary constants.**

By n independent constants, we mean to say that the most general solution of the DE cannot be expressed in fewer than n constants. As an example, the second order DE

$$\frac{d^2y}{dx^2} + y = 0$$

has its most general solution of the form

$$y = A\cos x + B\sin x. \qquad \ldots(1)$$

(verify that this is a solution by explicit substitution).

Thus, two arbitrary and independent constants must be included in the general solution. We cannot reduce(1) to a relation containing only one arbitrary constant. On the other hand, it can be verified that the function

$$y = ae^{x+b}$$

is a solution to the second-order DE

$$\frac{d^2y}{dx^2} = y$$

but even through it (seems to)contain two arbitrary constants, it is not the general solution to this DE. This is because it can be reduced to a relation involving only one arbitrary constant :

$$y = ae^{x+b}$$

$$= ae^x \cdot e^b$$

$$= ce^x \qquad \text{(where } c = a \cdot e^b\text{)}$$

Let us summarise what we've seen till now : the most general solution of an n^{th} order DE will consist of n arbitrary constants; conversely, from a functional relation involving n arbitrary constants, an n^{th} order DE can be generated (we'll soon see how to do this). We are generally interested in solutions of the DE satisfying some particular constraints (say, some initial values).Since the most general solution of the DE involves n arbitrary constant, we see that the maximum member of independent conditions which can be imposed on a solution of the DE is n.

As a first example, consider the functional relation

$$y = x^2 + c_1e^{2x} + c_2e^{3x} \qquad \text{...(1)}$$

This curve's equation contains two arbitrary constants; as we vary c_1 and c_2, we obtain different curves; those curves constitute a family of curves. All members of this family will satisfy the DE that we can generate from this general relation; this DE will be second order since the relation contains two arbitrary constants.

We now see how to generate the DE. Differentiate the given relation twice to obtain

$$y' = 2x + 2c_1e^{2x} + 3c_2e^{3x} \qquad \text{...(2)}$$

$$y'' = 2 + 4c_1e^{2x} + 9c_2e^{3x} \qquad \text{...(3)}$$

From (1), (2) and (3), c_1 and c_2 can be eliminated to obtain

$$\begin{vmatrix} e^{2x} & e^{3x} & x^2 - y \\ 2e^{2x} & 3e^{3x} & 2x - y' \\ 4e^{2x} & 9e^{3x} & 2 - y'' \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1 & 1 & x^2 - y \\ 2 & 3 & 2x - y' \\ 4 & 9 & 2 - y'' \end{vmatrix} = 0$$

$$\Rightarrow \quad 6 - 3y'' - 18x + 9y' + 8x - 4y' - 4 + 2y'' + 6x^2 - 6y = 0$$

$$\Rightarrow \quad y'' - 5y' + 6y = 6x^2 - 10x + 2 \qquad \text{...(4)}$$

This is the required DE; it corresponds to the family of curves given by (1). Differently put, the most general solution of this DE is given by (1).

As an exercise for the reader, show that the DE corresponding to the general equation

$$y = Ae^{2x} + Be^{x} + C$$

where A, B, C are arbitrary constants, is

$$y''' - 3y'' + 2y' = 0$$

As expected, the three arbitrary constants cause the DE to be third order.

Example – 1

Find the DE corresponding to the family of rectangular hyperbolas $xy = c^2$.

Solution: Since the equation for a rectangular hyperbola contains only one arbitrary constants, the corresponding DE for the family of rectangular hyperbolas will be first order and can be obtained by differentiation once.

$$xy = c^2$$

$$\Rightarrow \quad xy' + y = 0$$

This is the required DE. ◀

Example – 2

Find the DE associated with the family of circles of a fixed radius r.

Solution: The circles are of a fixed radius but their centres are not. Let the centre be denoted by the variable point (h, k).

Then the equation of an arbitrary circle of the family is

$$(x-h)^2 + (y-k)^2 = r^2 \qquad \text{...(1)}$$

This contains two arbitrary constants and therefore will give rise to a second-order DE.

Differentiating (1), we have

$$(x-h) + (y-k)\frac{dy}{dx} = 0 \qquad \text{...(2)}$$

Differentiating (2) again, we have

$$1+(y-k)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0 \qquad ...(3)$$

$$\Rightarrow \qquad (y-k)=-\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} \qquad ...(4)$$

Using (4) in (2), we have

$$(x-h)=\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\frac{dy}{dx}}{\frac{d^2y}{dx^2}} \qquad ...(5)$$

Using (4) and (5) in (1), and simplifying, we have the required DE as

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=r^2\left(\frac{d^2y}{dx^2}\right)^2$$

which as expected, is second order. ◀

If all this talk about arbitrary constants and solutions to DEs, confuses you, lets view the whole discussion from a slightly different perspective.

Referring to example -1, suppose we are given the DE

$$xy'+y=0 \qquad ...(1)$$

This is a first-order DE and its most general solution will contain one arbitrary constant; in fact, the most general solution of this DE is

$$xy=\lambda$$

where λ is an arbitrary constant. Now suppose we are told that the curve satisfying the DE in (1) passes through (2, 2). This additional constraint enables us to determine the value of λ for this particular curve :

$$(2)(2)=\lambda \qquad \Rightarrow \qquad \lambda=4$$

and thus the curve $xy=4$ is a particular solution of the DE in (1).

To emphasize once more,

$xy=\lambda$ is a general solution to (1)

while $xy=4$ is a particular solution to (1)

which was obtained from the general solution by using the fact that the curve passes through (2, 2).

As another example, consider the DE obtained in example - 2.

$$\left(1+(y')^2\right)^3 = r^2 (y'')^2$$

This is a second-order DE and its most general solution will contain two arbitrary constants; the most general solution can be found to be

$$(x-\alpha)^2 + (y-\beta)^2 = r^2$$

where α and β are arbitrary constants .

To determine a particular solution to this DE, we need **two** additional constraints which can enable us to evaluate α and β.

Example – 3

Find the DE associated with the family of straight lines, each of which is at a constant distance *p* from the origin.

Solution: Any such line has the equation

$$x\cos\alpha + y\sin\alpha = p \qquad ...(1)$$

where α is a variable. Different values of α give different lines belonging to this family. Since the equation representing this family contains only one arbitrary constant, its corresponding DE will be first order .

Differentiating(1), we have

$$\cos\alpha + y'\sin\alpha = 0$$

$$\Rightarrow \qquad \tan\alpha = -\frac{1}{y'}$$

$$\Rightarrow \qquad \sin\alpha = \frac{-1}{\sqrt{1+(y')^2}}, \quad \cos\alpha = \frac{y'}{\sqrt{1+(y')^2}} \qquad ...(2)$$

Using (2) in (1), we have

$$\frac{xy'}{\sqrt{1+(y')^2}} - \frac{y}{\sqrt{1+(y')^2}} = p$$

$$\Rightarrow \qquad (xy'-y)^2 = p^2(1+(y')^2)$$

As expected, this is a first order DE. ◀

TRY YOURSELF - I

Q. 1 Find the DE of all circles in the first quadrant which touch the coordinate axes.

Q. 2 Find the DE of all circles touching the x-axis at the origin.

Q. 3 Find the DE of all hyperbolas having their axes along the coordinate axes.

Section - 2 SOLVING DIFFERENTIAL EQUATIONS

In this section, we consider how to evaluate the general solution of a DE. You must appreciate the fact that evaluating the general solution of an arbitrary DE is not a simple task, in general. Over time, many methods have been developed to solve particular classes of DEs. Fortunately for us, at this level we are required to deal with only the simplest of cases.

We'll be considering only first order and first degree DEs. Note that any such DE can be written in the general form

$$\boxed{M(x, y)dx + N(x, y)dy = 0}$$

where M and N are functions of x and y.

TYPE - 1 : VARIABLE SEPARABLE FORM

This is by and large the simplest type of DE that we'll encounter. As the name suggests, in such an equation, M is a function of x only and N is a function of y only. Thus, such a DE is of the form

$$f(x)dx + g(y)dy = 0$$

which can be **solved** by straightforward integration to obtain

$$\int f(x)dx + \int g(y)dy = C$$

where C is an arbitrary constant.

Observe how the "variables are separated" in this type of DE and its general solution.

As a simple example, consider the DE

$$xdx + y^2dy = 0$$

This is obviously in variable -separable form. Integrating, we obtain

$$\int xdx + \int y^2dy = C$$

$$\Rightarrow \quad \frac{x^2}{2} + \frac{y^3}{3} = C$$

This is the required general solution of the DE.

Example – 4

Solve the DE $y - x\dfrac{dy}{dx} = \lambda\left(y^2 + \dfrac{dy}{dx}\right)$.

Solution: A little observation will show you that the variables are separable in this DE:

$$y(1-\lambda y) = \frac{dy}{dx}(x+\lambda)$$

$$\Rightarrow \quad \frac{dy}{y(1-\lambda y)} = \frac{dx}{x+\lambda}$$

$$\Rightarrow \quad \left(\frac{1}{y} + \frac{\lambda}{1-\lambda y}\right)dy = \frac{dx}{x+\lambda}$$

Integrating both sides, we have

$$\ln y - \ln(1-\lambda y) = \ln(x+\lambda) + C$$

$$\Rightarrow \quad \ln\left(\frac{y}{1-\lambda y}\right) = \ln(x+\lambda) + \ln C'$$

In the last step, we have written the arbitrary constant of integration C as C' so that the whole expression can be combined now by taking antilog on both sides. (There's no loss of generality in doing so and it is often done to make the final expression look simpler). Thus, we now have,

$$\frac{y}{1-\lambda y} = C'(x+\lambda)$$

$$\Rightarrow \quad y = C'(x+\lambda)(1-\lambda y)$$

This is the required general solution of the DE; as expected it contains only one arbitrary constant. ◀

Example – 5

Solve the DE $xy^2 \frac{dy}{dx} = 1 - x^2 + y^2 - x^2y^2$.

Solution: Again, this DE is of the variable separable form as can be made evident by a slight rearrangement.

$$xy^2 \frac{dy}{dx} = (1-x^2)(1+y^2)$$

$$\Rightarrow \quad \left(\frac{y^2}{1+y^2}\right)dy = \left(\frac{1-x^2}{x}\right)dx$$

$$\Rightarrow \quad \left(1 - \frac{1}{1+y^2}\right)dy = \left(\frac{1}{x} - x\right)dx$$

Integrating both sides, we have

$$y - \tan^{-1} y = \ln x - \frac{x^2}{2} + C$$

This is the required general solution. ◀

Sometimes, the DE might not be in the variable-separable (VS) form; however, some manipulations might be able to transform it to a VS form. Lets see how this can be done. Consider the DE

$$\frac{dy}{dx} = \cos(x+y)$$

This is obviously not in VS form. Observe what happens if we use the following substitution in this DE:

$$x + y = v$$

$$\Rightarrow \quad 1 + \frac{dy}{dx} = \frac{dv}{dx}$$

Thus, the DE transforms to

$$\frac{dv}{dx} - 1 = \cos v$$

$$\Rightarrow \quad \frac{dv}{dx} = 1 + \cos v$$

$$\Rightarrow \quad \frac{dv}{1+\cos v} = dx$$

which is clearly a VS form. Integrating both sides, we obtain

$$\int \frac{dv}{1+\cos v} = \int dx$$

$$\Rightarrow \quad \frac{1}{2}\int \sec^2 \frac{v}{2} dv = \int dx$$

$$\Rightarrow \quad \tan\frac{v}{2} = x + C$$

$$\Rightarrow \quad \tan\left(\frac{x+y}{2}\right) = x + C$$

This is the required general solution to the DE.

From this example, you might be able to infer that any DE of the form

$$\frac{dy}{dx} = f(ax + by + c)$$

is reducible to a VS form using the technique described. Let us confirm this explicitly:

Substitute

$$ax + by + c = v$$

$$\Rightarrow \quad a + b\frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{b}\left(\frac{dv}{dx} - a\right)$$

Thus, our DE reduces to

$$\frac{1}{b}\left(\frac{dv}{dx} - a\right) = f(v)$$

$$\Rightarrow \quad \frac{dv}{dx} = a + bf(v)$$

$$\Rightarrow \quad \frac{dv}{a + bf(v)} = dx$$

which is obviously in VS form, and hence can be solved.

Example – 6

Solve the DE $\frac{dy}{dx} = \frac{r^2}{(x+y)^2}$.

Solution: Substituting $x + y = v,$ we have

$$\frac{dy}{dx} = \frac{dv}{dx} - 1$$

and thus the DE reduces to

$$\frac{dv}{dx} - 1 = \frac{r^2}{v^2}$$

$$\Rightarrow \quad \frac{v^2}{r^2 + v^2}dv = dx$$

$$\Rightarrow \quad \left(1 - \frac{r^2}{r^2 + v^2}\right)dv = dx$$

Integrating, we have

$$v - r\tan^{-1}\left(\frac{v}{r}\right) = x + C$$

$$\Rightarrow \qquad (x+y) - r\tan^{-1}\left(\frac{x+y}{r}\right) = x + C$$ ◀

Example – 7

Solve the DE $\dfrac{dy}{dx} = \dfrac{(x+y)+(x+y-1)\ln(x+y)}{\ln(x+y)}$.

Solution: Again, the substitution $x+y=v$ will reduce this DE to the following VS form:

$$\frac{dv}{dx} - 1 = \frac{v+(v-1)\ln v}{\ln v}$$

$$= (v-1) + \frac{v}{\ln v}$$

$$\Rightarrow \qquad \frac{dv}{dx} = v + \frac{v}{\ln v}$$

$$\Rightarrow \qquad \frac{\ln v}{v(1+\ln v)}dv = dx$$

Integrating, we have

$$\int \frac{\ln v}{v(1+\ln v)}dv = \int dx$$

To evaluate the integral on the LHS, we use the substitution $(1+\ln v)=t$ which gives $\dfrac{1}{v}dv = dt$.

Thus,

$$\int \frac{t-1}{t}dt = \int dx$$

$$\Rightarrow \qquad t - \ln t = x + C$$

$$\Rightarrow \qquad (1+\ln v) - \ln(1+\ln v) = x + C$$

$$\Rightarrow \qquad \big(1+\ln(x+y)\big) - \ln\big(1+\ln(x+y)\big) = x + C$$ ◀

TYPE - 2 : HOMOGENEOUS DEs

By definition, a homogeneous function $f(x, y)$ of degree *n* satisfies the property

$$f(\lambda x, \lambda y) = \lambda^n f(x, y)$$

For example, the functions

$$f_1(x, y) = x^3 + y^3$$

$$f_2(x, y) = x^2 + xy + y^2$$

$$f_3(x, y) = x^3 e^{x/y} + xy^2$$

are all homogeneous functions, of degrees three, two and three respectively (verify this assertion).

Observe that any homogeneous function $f(x, y)$ of degree *n* can be equivalently written as follows:

$$f(x, y) = x^n f\left(\frac{y}{x}\right) = y^n f\left(\frac{x}{y}\right)$$

For example,

$$f_1(x, y) = x^3 + y^3$$

$$= x^3\left(1 + \left(\frac{y}{x}\right)^3\right) = y^3\left(1 + \left(\frac{x}{y}\right)^3\right)$$

Having seen homogeneous functions we define homogeneous DEs as follows :

> Any DE of the form $M(x, y)\, dx + N(x, y)\, dy = 0$ or $\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}$ is called homogeneous if $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree.

What is so special about homogeneous DEs ? Well, it turns out that they are extremely simple to solve. To see how, we express both $M(x, y)$ and $N(x, y)$ as, say $x^n M\left(\frac{y}{x}\right)$ and $x^n N\left(\frac{y}{x}\right)$. This can be done since $M(x, y)$ and $N(x, y)$ are both homogeneous functions of degree *n*. Doing this reduces our DE to

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)} = -\frac{x^n M\left(\frac{y}{x}\right)}{x^n N\left(\frac{y}{x}\right)} = -\frac{M\left(\frac{y}{x}\right)}{N\left(\frac{y}{x}\right)} = P\left(\frac{y}{x}\right)$$

(The function $P(t)$ stands for $\frac{-M(t)}{N(t)}$)

Now, the simple substitution $y = vx$ reduces this DE to a VS form :

$$y = vx$$

$$\Rightarrow \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Thus, $\frac{dy}{dx} = P\left(\frac{y}{x}\right)$ transforms to

$$v + x\frac{dv}{dx} = P(v)$$

$$\Rightarrow \qquad \frac{dv}{P(v) - v} = \frac{dx}{x}$$

This can now be integrated directly since it is in VS form.

Let us see some examples of solving homogeneous DEs.

Example – 8

Solve the DE $\frac{dy}{dx} = \frac{2x - y}{x + y}$.

Solution: This is obviously a homogeneous DE of degree one since the RHS can be written as

$$\frac{2x - y}{x + y} = \frac{x \cdot \left(2 - \frac{y}{x}\right)}{x \cdot \left(1 + \frac{y}{x}\right)} = \frac{2 - \frac{y}{x}}{1 + \frac{y}{x}}$$

Using the substitution $y = vx$ reduces this DE to

$$v + x\frac{dv}{dx} = \frac{2 - v}{1 + v}$$

$$\Rightarrow \qquad x\frac{dv}{dx} = \frac{2 - v}{1 + v} - v$$

$$= \frac{2 - 2v - v^2}{1 + v}$$

$$= \frac{3 - (1 + v)^2}{1 + v}$$

$$\Rightarrow \qquad \frac{(1 + v)}{3 - (1 + v)^2}dv = \frac{dx}{x}$$

Using $t = 1 + v$ above, we have

$$\frac{t}{3-t^2}\,dt = \frac{dx}{x}$$

Integrating, we have

$$\int \frac{t}{3-t^2}\,dt = \int \frac{dx}{x}$$

$$\Rightarrow \qquad -\frac{1}{2}\ln\left|3-t^2\right| = \ln x + \ln C_1$$

$$\Rightarrow \qquad \ln\left(x^2\left(3-t^2\right)\right) = C_2$$

$$\Rightarrow \qquad x^2\left(3-t^2\right) = C$$

$$\Rightarrow \qquad x^2\left(3-(1+v)^2\right) = C$$

$$\Rightarrow \qquad x^2\left(2-2v-v^2\right) = C$$

Substituting $\frac{y}{x}$ for v, we finally obtain the required general solution to the DE:

$$2x^2 - 2xy - y^2 = C.$$ ◀

Example – 9

Solve the DE $xdy - ydx = \sqrt{x^2+y^2}\,dx.$

Solution: Upon rearrangement, we have

$$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$$

$$= \left(\frac{y}{x}\right) + \sqrt{1+\left(\frac{y}{x}\right)^2}$$

This is obviously a first degree homogeneous DE. We substitute $y = vx$ to obtain:

$$v + x\frac{dv}{dx} = v + \sqrt{1+v^2}$$

$$\Rightarrow \qquad \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

Integrating both sides, we have

$$\ln\left|v+\sqrt{1+v^2}\right| = \ln x + \ln C$$

$$= \ln Cx$$

$$\Rightarrow \quad \frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}} = Cx$$

◀

Example – 10

Solve the DE $\left(1+e^{x/y}\right)dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0.$

Solution: This DE can be rearranged as

$$\frac{dx}{dy} = \frac{e^{x/y}\left(\frac{x}{y}-1\right)}{e^{x/y}+1}$$

Using the substituting $x = vy$ (note : not $y = vx$) can reduce this DE to a VS form. (We did not use $y = vx$ since that would've led to an expression involving complicated exponentials).

We now have

$$v + y\frac{dv}{dy} = \frac{e^v(v-1)}{e^v+1}$$

$$\Rightarrow \quad \frac{dy}{y} = -\frac{e^v+1}{e^v+v}dv$$

Integrating both sides, we have

$$\ln y = -\ln\left|e^v+v\right| + \ln C$$

$$\Rightarrow \quad y(e^v+v) = C$$

$$\Rightarrow \quad e^{x/y}+\frac{x}{y} = \frac{C}{y}$$

◀

This example should serve the show that $y = vx$ will not always be the most appropriate substitution to solve a homogeneous DE; $x = vy$ could be more appropriate in such a scenario, as in the example above.

Many a times, the DE specified may not be homogeneous but some suitable manipulation might reduce it to a homogeneous form. Generally, such equations involve a function of a rational expression whose numerator and denominator are linear functions of the variable, i.e., of the form

$$\frac{dy}{dx} = f\left(\frac{ax+by+c}{dx+cy+f}\right) \qquad ...(1)$$

Note that the presence of the constant *c* and *f* causes this DE to be non-homogeneous.

To make it homogeneous, we use the substitutions

$$x \to X + h$$

$$y \to Y + k$$

and select *h* and *k* so that

$$\left.\begin{aligned} ah+bk+c &= 0 \\ dh+ek+f &= 0 \end{aligned}\right\} \qquad ...(2)$$

This can always be done $\left(\text{if } \frac{a}{b} \neq \frac{d}{e}\right)$. The RHS of the DE in (1) now reduces to

$$f\left(\frac{a(X+h)+b(Y+k)+c}{d(X+h)+e(Y+k)+f}\right)$$

$$= f\left(\frac{aX+bY+(ah+bk+c)}{dX+eY+(dh+ek+f)}\right)$$

$$= f\left(\frac{aX+bY}{dX+eY}\right) \qquad \text{(Using (2))}$$

This expression is clearly homogeneous ! The LHS of (1) is $\frac{dy}{dx}$ which equals $\frac{dy}{dY}\cdot\frac{dY}{dX}\cdot\frac{dX}{dx}$. Since $\frac{dy}{dY}\cdot\frac{dx}{dX}=1$, the LHS $\frac{dy}{dx}$ equals $\frac{dY}{dX}$. Thus, our equation becomes

$$\frac{dY}{dX} = f\left(\frac{aX+bY}{dX+eY}\right) \qquad ...\ (3)$$

We have thus succeeded in transforming the non-homogeneous DE in (1) to the homogeneous DE in (3). This can now be solved as described earlier.

Let us apply this technique in some examples. ◀

Example – 11

Solve the DE $\frac{dy}{dx}=\frac{2y-x-4}{y-3x+3}$.

Solution: We substitute $x \to X+h$ and $y \to Y+k$ where h, k need to be determined :

$$\frac{dy}{dx}=\frac{dY}{dX}=\frac{(2Y-X)+(2k-h-4)}{(Y-3X)+(k-3h+3)}$$

h and k must be chosen so that

$$2k-h-4=0$$

$$k-3h+3=0$$

This gives $h=2$ and $k=3$. Thus,

$$x=X+2$$

$$y=Y+3$$

Our DE now reduces to

$$\frac{dY}{dX}=\frac{2Y-X}{Y-3X}$$

Using the substitution $Y=vX$, and simplifying, we have (verify),

$$\frac{v-3}{v^2-5v+1}dv=\frac{-dX}{X}$$

We now integrate this DE which is VS; the left-hand side can be integrated by the techniques described in the unit on Indefinite Integration.

Finally, we substitute $v=\frac{Y}{X}$ and

$$X=x-2$$

$$Y=y-3$$

to obtain the general solution. ◀

Suppose our DE is of the form

$$\frac{dy}{dx}=f\left(\frac{ax+by+c}{dx+ey+f}\right)$$

We try to find h, k so that

$$ah+bk+c=0$$

$$dh+ek+f=0$$

What if this system does not yield a solution ? Recall that this will happen if $\frac{a}{b} = \frac{d}{e}$. How do we reduce the DE to a homogeneous one in such a case ?

Let $\frac{a}{d} = \frac{b}{e} = \lambda$ (say). Thus,

$$\frac{ax+by+c}{dx+ey+f} = \frac{\lambda(dx+ey)+c}{dx+ey+f}$$

This suggests the substitution $dx+ey=v,$ which'll give

$$d + e\frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{e}\left(\frac{dv}{dx} - d\right)$$

Thus, our DE reduces to

$$\frac{1}{e}\left(\frac{dv}{dx} - d\right) = \frac{\lambda v + c}{v+f}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{\lambda ev + ec}{v+f} + d$$

$$= \frac{(\lambda e + d)v + (ec+df)}{v+f}$$

$$\Rightarrow \quad \frac{(v+f)}{(\lambda e+d)v + ec + df}\,dv = dx$$

which is in VS form and hence can be solved. ◀

Example – 12

Solve the DE $\frac{dy}{dx} = \frac{x+2y-1}{x+2y+1}$.

Solution: Note that h, k do not exist in this case which can reduce this DE to homogeneous form. Thus, we use the substitution

$$x + 2y = v$$

$$\Rightarrow \quad 1 + 2\frac{dy}{dx} = \frac{dv}{dx}$$

Thus, our DE becomes

$$\frac{1}{2}\left(\frac{dv}{dx}-1\right)=\frac{v-1}{v+1}$$

$$\Rightarrow \quad \frac{dv}{dx}=\frac{2v-2}{v+1}+1$$

$$=\frac{3v-1}{v+1}$$

$$\Rightarrow \quad \frac{v+1}{3v-1}dv=dx$$

$$\Rightarrow \quad \frac{1}{3}\left(1+\frac{4}{3v-1}\right)dv=dx$$

Integrating, we have

$$\frac{1}{3}\left(v+\frac{4}{3}\ln(3v-1)\right)=x+C_1$$

Substituting $v=x+2y,$ we have

$$x+2y+\frac{4}{3}\ln(3x+6y-1)=3x+C_2$$

$$\Rightarrow \quad y-x+\frac{2}{3}\ln(3x+6y-1)=C$$

◀

TYPE - 3 **FIRST ORDER DEs**

We now come to a very important class of DEs : first-order linear DEs, their importance arising from the fact that many natural phenomena can be described using such DEs.

First order linear DEs take the form

$$\frac{dy}{dx}+P(x)y=Q(x)$$

where P and Q are functions of x alone.

To solve such DEs, the method followed is as described below :

We multiply both sides of the DE by a quantity called the integrating factor (I.F.) where

$$I.F.=e^{\int P dx}$$

Why this is chosen as the I.F. will soon become clear when we see what the I. F. actually does :

$$e^{\int Pdx}\left(\frac{dy}{dx}+Py\right)=e^{\int Pdx}\cdot Q$$

The left hand side now becomes exact, in the sense that it can be expressed as the exact differential of some expression :

$$e^{\int Pdx}\left(\frac{dy}{dx}+Py\right)=\frac{d}{dx}\left(ye^{\int Pdx}\right)$$

Now our DE becomes

$$\frac{d}{dx}\left(ye^{\int Pdx}\right)=Q\cdot e^{\int Pdx}$$

This can now easily be integrated to yield the required general solution:

$$ye^{\int Pdx}=\int\left(Qe^{\int Pdx}\right)dx+C$$

You are urged to re-read this discussion until you fully understand its significance. In particular, you must understand why multiplying the DE by the I. F. $e^{\int Pdx}$ on both sides reduces its left hand side to an exact differential. ◀

Example – 13

Solve the DE $\frac{dy}{dx}+y\tan x=\cos x.$

Solution: Comparing this DE with the standard form of the linear DE $\frac{dy}{dx}+Py=Q,$ we see that

$$P(x)=\tan x,\quad Q(x)=\cos x$$

Thus, the I.F. is

$$I.F.=e^{\int \tan x dx}=e^{\ln(\sec x)}=\sec x$$

Multiplying by $\sec x$ on both sides of the given DE, we obtain

$$\sec x\frac{dy}{dx}+y\tan x\sec x=1$$

The left hand side is an exact differential :

$$\frac{d}{dx}(y\sec x) = 1$$

$$\Rightarrow \quad d\left(y\sec x\right) = dx$$

Integrating both sides, we obtain the solution to our DE as

$$y\sec x = x + C$$ ◄

Example – 14

Solve the DE $\frac{dx}{dy} + \frac{x}{y} = y^2$.

Solution: The I.F. is

$$I.F. = e^{\int P(y)dy} \qquad \text{(}y\text{ is the 'independent' variable in this DE)}$$

$$= e^{\int \frac{1}{y}dy}$$

$$= e^{\ln y}$$

$$= y$$

Multiplying by the I.F. on both sides, we have

$$y\frac{dx}{dy} + x = y^3$$

$$\Rightarrow \quad \frac{d}{dy}(xy) = y^3$$

$$\Rightarrow \quad d(xy) = y^3 dy$$

Integrating both sides gives

$$xy = \frac{y^4}{4} + C$$ ◄

Example – 15

Solve the DE $\frac{dy}{dx} = x^3y^3 - xy$.

Solution: We have,

$$\frac{dy}{dx} + xy = x^3 y^3$$

Note that since the RHS contains the term y^3, this DE is not in the standard linear DE form. However, a little artifice can enable us to reduce this to the standard form.

Divide both sides of the equation by y^3.

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2} = x^3 \qquad ...(1)$$

Substitute $$\frac{1}{y^2} = v$$

$$\Rightarrow \quad \frac{-2}{y^3}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \quad \frac{1}{y^3}\frac{dy}{dx} = \frac{-1}{2}\frac{dv}{dx} \qquad ...(2)$$

Using (2) in (1), we have

$$\frac{-1}{2}\frac{dv}{dx} + xv = x^3$$

$$\Rightarrow \quad \frac{dv}{dx} + (-2x)v = -2x^3 \qquad ...(3)$$

This is now in the standard first-order linear DE form. The I.F. is

$$I.F. = e^{\int -2x dx} = e^{-x^2}$$

Thus, the solution to (3) is

$$v \times I.F. = \int Q(x) \times (I.F) dx$$

$$\Rightarrow \quad ve^{-x^2} = -2\int x^3 e^{-x^2} dx$$

Performing the integration on the RHS by the substitution $t = -x^2$ and then using integration by parts, we obtain

$$ve^{-x^2} = e^{-x^2}(x^2+1) + C$$

$$\Rightarrow \quad \frac{1}{y^2}e^{-x^2} = e^{-x^2}(x^2+1) + C$$

This is the required general solution to the DE. ◀

This example also tells us how to solve a DE of the general form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \qquad \text{...(4)}$$

We divide by y^n on both sides :

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)y^{-n+1} = Q(x)$$

and then substitute $y^{-n+1} = v$ and proceed as described in the solution above.

DEs that take the form in (4) are known as Bernoulli's DEs.

EXACT DEs*

In the last section, we discussed how the multiplication of the $I.F. = e^{\int P dx}$ on both sides of the linear DE

$$\frac{dy}{dx} + P(x)y = Q(x)$$

renders this into an exact DE. We now consider the general case of exact DEs. In particular, we want to see what condition must be satisfied in order that the DE

$$M(x, y)dx + N(x, y)dy = 0 \qquad ...(1)$$

is exact.

In order for this DE to be exact, it's LHS must be expressible as the complete differential of some function $f(x, y)$, i.e.

$$M(x, y)dx + N(x, y)dy = d\left(f(x, y)\right) \qquad ...(2)$$

Now since the function $f(x, y)$ is a function of both *x* and *y*, its total differential is a sum of partical differentials with respect to *x* and *y*, i.e.,

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \qquad ...(3)$$

Comparing (1) and (2), we have

$$\frac{\partial f}{\partial x} = M, \qquad \frac{\partial f}{\partial y} = N \qquad ...(4)$$

This gives

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial M}{\partial y}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

For continuous $f(x, y)$,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

and thus for the DE in (1) to be exact, we see that the necessary (and in fact sufficient) condition is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

* **This section contains advanced material and is optional.**

If this condition is satisfied, the DE in (1) reduces to

$$df(x, y) = 0$$

which upon integration leads to the required solution:

$$f(x, y) = C$$

As an example, consider the DE

$$3x(xy-2)dx + (x^3 + 2y)dy = 0 \qquad \text{...(5)}$$

We have,

$$M(x, y) = 3x(xy-2) \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 3x^2$$

$$N(x, y) = x^3 + 2y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 3x^2$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the DE is exact and hence we can find a function $f(x, y)$ such that the DE is expressible as $df(x, y) = 0$. Let us try to explicitly find this function.

From (4), we have

$$\frac{\partial f}{\partial x} = M \quad \Rightarrow \quad \frac{\partial f}{\partial x} = 3x(xy-2)$$

Integrating with respect to *x*, while treating *y* as a constant, we have

$$f(x, y) = x^3 y - 3x^2 + \phi(y) \qquad \text{...(6)}$$

The function $\phi(y)$ acts as the arbitrary constant of integration, since *y* is constant for this integration process.

From (4) again we have

$$\frac{\partial f}{\partial y} = N \quad \Rightarrow \quad \frac{\partial f}{\partial y} = x^3 + 2y$$

Evaluating $\frac{\partial f}{\partial y}$ from (6), we have

$$x^3 + \phi'(y) = \frac{\partial f}{\partial y} = x^3 + 2y$$

$$\Rightarrow \quad \phi'(y) = 2y$$

$$\Rightarrow \quad \phi(y) = y^2 + C \qquad \text{...(7)}$$

Finally substituting (7) into (6), we have

$$f(x,y) = x^3y - 3x^2 + y^2 + C$$

Thus, the solution to the DE in (5) is

$$f(x,y) = \text{constant}$$

$$\Rightarrow \quad x^3y - 3x^2 + y^2 = \text{constant} \qquad ...(8)$$

You are urged to verify that (8) is indeed the required solution by differentiating (8) and observing that (5) is obtained.

Example – 16

Solve the DE $2xydx + (x^2 + 3y^2)dy = 0$.

Solution: First of all, notice that this DE is homogeneous :

$$\frac{dy}{dx} = -\frac{2xy}{x^2 + 3y^2}$$

The substitution $y = vx$ leads to

$$v + x\frac{dv}{dx} = \frac{-2vx^2}{x^2 + 3v^2x^2} = \frac{-2v}{1 + 3v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -v - \frac{2v}{1 + 3v^2}$$

$$= \frac{-3v - 3v^3}{1 + 3v^2}$$

$$= \frac{-3v(1 + v^2)}{1 + 3v^2}$$

$$\Rightarrow \quad \frac{1 + 3v^2}{v(1 + v^2)}dv = -3\frac{dx}{x}$$

The substitution $v^2 = t$ leads to

$$\left(\frac{1 + 3t}{1 + t} \cdot \frac{1}{2t}\right)dt = -3\frac{dx}{x}$$

$$\Rightarrow \quad \left(\frac{1}{1 + t} + \frac{1}{2t}\right)dt = -3\frac{dx}{x}$$

Integrating both sides gives

$$\ln(1+t)+\frac{1}{2}\ln t=-3\ln x+\ln C$$

$$\Rightarrow \qquad \sqrt{t}(1+t)x^3=C$$

$$\Rightarrow \qquad v(1+v^2)x^3=C$$

$$\Rightarrow \qquad y(x^2+y^2)=C$$

We now solve this DE again using the exact differential approach since by observation this DE satisfies the required criterion for it to be exact.

We have,

$$\frac{\partial f}{\partial x}=M=2xy, \qquad \frac{\partial f}{\partial y}=N=x^2+3y^2$$

Integrating the first relation, we have

$$f(x,y)=x^2y+\phi(y) \qquad ...(1)$$

Differentiating this w.r.t. y and comparing with the expression for $\frac{\partial f}{\partial y}$ above, we have

$$\frac{\partial f}{\partial y}=x^2+\phi'(y)=x^2+3y^2$$

$$\Rightarrow \qquad \phi'(y)=3y^2$$

$$\Rightarrow \qquad \phi(y)=y^3+C' \qquad ...(2)$$

Thus, from (1) and (2),

$$f(x,y)=x^2y+y^3+C'$$

The solution to the exact DE is

$$f(x,y)=\text{constant}$$

$$\Rightarrow \qquad x^2y+y^3=C$$

$$\Rightarrow \qquad y(x^2+y^2)=C$$

which is the same as the one obtained earlier. Thus, the exact differential approach might lead to the solution faster than the other approaches we've discussed earlier. ◀

Sometimes, the fact that the DE is exact is evident merely be inspection. We list down such exact differentials (verify the truth of these relations):

$xdy + ydx$	$\longrightarrow$	$d(xy)$
$\dfrac{xdy - ydx}{x^2}$	$\longrightarrow$	$d\left(\dfrac{y}{x}\right)$
$\dfrac{ydx - xdy}{y^2}$	$\longrightarrow$	$d\left(\dfrac{x}{y}\right)$
$\dfrac{xdy - ydx}{x^2 + y^2}$	$\longrightarrow$	$d\left(\tan^{-1}\dfrac{y}{x}\right)$
$xdx + ydy$	$\longrightarrow$	$d\left(\dfrac{x^2 + y^2}{2}\right)$

Table -1

Example – 17

Solve the DE $\dfrac{xdy}{x^2 + y^2} = \left(\dfrac{y}{x^2 + y^2} - 1\right)dx.$

Solution: Upon rearrangement, this DE gives

$$\frac{xdy - ydx}{x^2 + y^2} = -dx \qquad ...(1)$$

From Table-1, the L.H.S of (1) is the exact differential $d\left(\tan^{-1}\dfrac{y}{x}\right)$. Thus, our DE reduces to $d\left(\tan^{-1}\dfrac{y}{x}\right) + dx = 0$

Integrating, we obtain the solution as

$$\tan^{-1}\frac{y}{x} + x = C$$

However, it is very likely that we won't be able to make out just be inspection whether the DE is exact or not.

If the DE is not exact, it can be rendered exact by multiplying it with an integrating factor I.F. In the case of the first-order linear DE

$$\frac{dy}{dx} + Py = Q$$

the I.F. $e^{\int P dx}$ renders the DE exact:

$$\frac{d}{dx}\left(ye^{\int P dx}\right) = Qe^{\int P dx}$$

and the solution is now obtainable by integration.

If fact, a systematic approach exists to determine the I.F. in a general case (if such an I.F. is possible at all.). However, we'll not be discussing that approach here since it is beyond our current scope.

Let us see another example, where the solution is easily obtained by the recognition of exact differentials present in the equation. ◀

Example – 18

Solve the DE $x\cos\left(\frac{y}{x}\right)(ydx + xdy) = y\sin\left(\frac{y}{x}\right)(xdy - ydx)$.

Solution: Upon rearrangement, we have

$$ydx + xdy = \frac{y}{x}\tan\frac{y}{x}(xdy - ydx)$$

$$= xy\tan\frac{y}{x}\left(\frac{xdy - ydx}{x^2}\right)$$

$$\Rightarrow \qquad \frac{ydx + xdy}{xy} = \tan\frac{y}{x}\left(\frac{xdy - ydx}{x^2}\right)$$

From Table-1 this can be written as

$$\frac{d(xy)}{xy} = \tan\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right)$$

The solution is now obtained simply by integrating both sides :

$$\ln(xy) = \ln\left(\sec\left(\frac{y}{x}\right)\right) + \ln C$$

$$\Rightarrow \qquad xy = C\sec\left(\frac{y}{x}\right)$$ ◀

TRY YOURSELF - II

Solve the following DEs:

Q. 1 $xdx + ydy = a(x^2 + y^2)dy.$

Q. 2 $(3xy + y^2)dx + (x^2 + xy)dy = 0.$

Q. 3 $\frac{dy}{dx} = \frac{2x - y + 3}{x + 2y + 4}.$

Q. 4 $\frac{dy}{dx} + y\cot x = \cos x.$

Q. 5 $x\frac{dy}{dx} - y = y^3 \ln x.$

As described in the introduction, differential equations are so important for the very reason that they find a wide application in studying all sorts of scientific phenomena. The motion of a body in a force field, radioactive decay and population growth were some of the phenomena mentioned that must use DEs for analysis.

In some of the subsequent solved examples, we apply the DE- solving techniques that we've learnt in the previous section, to solve practical and interesting problems.

SOLVED EXAMPLES

Example – 1

Let $f:\mathbb{R}^+ \to \mathbb{R}$ be a differentiable function such that

$$f(x) = e - (x-1)\ln\frac{x}{e} + \int_1^x f(x)\,dx$$

Find a simple expression for *f*(*x*).

Solution: Differentiating the given relation, we have

$$\frac{df}{dx} = -\frac{(x-1)}{x} - \ln\frac{x}{e} + f$$

$$\Rightarrow \quad \frac{df}{dx} - f = \frac{1}{x} - \ln x$$

This is evidently a first-order linear DE; the IF is $e^{\int -dx} = e^{-x}$. Multiplying it across both sides of the DE renders the DE exact and its solution is given by

$$e^{-x}\cdot f = \int e^{-x}\left(\frac{1}{x} - \ln x\right)dx$$

$$= e^{-x}\ln x + C$$

$$\Rightarrow \quad f(x) = \ln x + Ce^{x} \qquad \text{... (1)}$$

From the relation specified in the equation, note that

$$f(1) = e - (1-1)\left(\ln\frac{1}{e}\right) + \int_1^1 f(x)\,dx$$

$$= e$$

From (1), $f(1) = Ce$. This gives $C = 1$.

Thus, the function *f*(*x*) has the simple form

$$f(x) = \ln x + e^{x}$$ ◀

Example – 2

Solve the following DEs

(a) $$\frac{dy}{dx}=\frac{\cos x\left(2\cos y-\sin^2 x\right)}{\sin y}$$

(b) $$\left(y\frac{dy}{dx}+2x\right)^2=\left(1+\left(\frac{dy}{dx}\right)^2\right)\left(y^2+2x^2\right)$$

Solution: **(a)** We have,

$$\sin y\frac{dy}{dx}-2\cos y\cos x=-\sin^2\cos x \qquad \text{... (1)}$$

Observe that the substitution $-\cos y=z$ will reduce this DE to a standard linear DE

$$-\cos y=z$$

$$\Rightarrow \quad \sin y\frac{dy}{dx}=\frac{dz}{dx} \qquad \text{...(2)}$$

Using (2) in (1), we have

$$\frac{dz}{dx}+\left(2\cos x\right)z=-\sin^2 x\cos x$$

The I.F for this DE is $e^{\int 2\cos x dx}=e^{2\sin x}$

Thus, the solution will be given by

$$ze^{2\sin x}=-\int e^{2\sin x}\cos x\cdot\sin^2 x\,dx \qquad \text{...(3)}$$

To integrate the R.H.S, we use the substitution $\sin x=t \Rightarrow \cos xdx=dt.$ Thus, the integral reduces to

$$-\int t^2e^{2t}\,dt$$

$$=-\frac{t^2e^{2t}}{2}-\frac{e^{2t}}{4}+\frac{te^{2t}}{2}+C' \qquad \text{(Integration by parts)}$$

$$=-\frac{\sin^2 x\cdot e^{2\sin x}}{2}-\frac{e^{2\sin x}}{4}+\frac{\sin x\cdot e^{2\sin x}}{2}+C'$$

Finally, the solution to the DE is, from (3)

$$z=-\cos y=-\frac{\sin^2 x}{2}-\frac{1}{4}+\frac{\sin x}{2}+C'e^{-2\sin x}$$

$$\Rightarrow \quad 4\cos y=2\sin^2 x-2\sin x+1+Ce^{-2\sin x}$$

◀

(b) Let $\frac{dy}{dx} = p$. Thus, this DE is

$$(py+2x)^2 = (1+p^2)(y^2+2x^2)$$

$$\Rightarrow \quad 2x^2p^2 - 4xyp + y^2 - 2x^2 = 0$$

$$\Rightarrow \quad p = \frac{4xy \pm \sqrt{16x^2y^2 - 8x^2(y^2-2x^2)}}{4x^2}$$

$$= \frac{4xy \pm \sqrt{8x^2y^2 + 16x^4}}{4x^2}$$

$$= \frac{4xy \pm 2\sqrt{2}\,x\sqrt{y^2+2x^2}}{4x^2}$$

$$= \frac{y}{x} \pm \frac{\sqrt{y^2+2x^2}}{\sqrt{2}x}$$

$$= \frac{y}{x} \pm \sqrt{\frac{1}{2}\left(\frac{y}{x}\right)^2 + 1}$$

The substitution $y = vx$ reduces this DE to

$$v + x\frac{dv}{dx} = v \pm \sqrt{\frac{v^2}{2} + 1}$$

$$\Rightarrow \quad \frac{dv}{\sqrt{v^2+2}} = \pm \frac{1}{\sqrt{2}} \frac{dx}{x}$$

$\Rightarrow$ Integrating both sides, we have

$$\ln\left|v + \sqrt{v^2+2}\right| = \pm \frac{1}{\sqrt{2}} \ln x + C'$$

$$\Rightarrow \quad \ln x \pm \sqrt{2} \ln\left|v + \sqrt{v^2+2}\right| = C$$

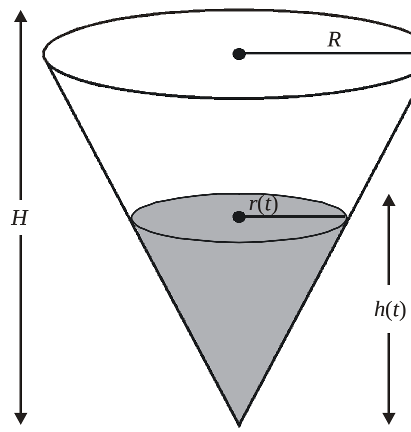
$$\Rightarrow \quad \ln x \pm \sqrt{2} \ln\left|\frac{y}{x} + \frac{\sqrt{v^2+2x^2}}{x}\right| = C \qquad \text{... (4)}$$

Thus, we obtain two different solutions to the DE, one corresponding to the " + " and one to the " – " sign in (4). ◀

Example – 3

A right circular cone with radius R and height H contains a liquid which evaporates at a rate proportional to its surface area in contact with air (proportionality constant = $k > 0$). Find the time after which the cone is empty.

Solution: We need to form a differential equation which describes the variation of the amount of water left in the cone with time.



Let us denote the height of the water remaining in the cup at time t by $h(t)$. Denote the volume at time t by $v(t)$

Fig - 1

From the geometry described in the figure above,

$$\frac{r(t)}{h(t)} = \frac{R}{H}$$

$$\Rightarrow \qquad h(t) = \frac{H}{R} r(t)$$

The volume $v(t)$ of the cone is

$$v(t) = \frac{1}{3}\pi r^2(t) h(t)$$

$$= \frac{\pi H}{3R} r^3(t) \qquad \text{.... (1)}$$

Now, it is specified that the rate of evaporation (the rate of decrease of the water's volume) is proportional to the surface area in contact with air:

$$\frac{dv}{dt} = -kr^2 \qquad \text{... (2)}$$

From (1) and (2), we have

$$\frac{H}{R} r^2 \frac{dr}{dt} = -kr^2$$

$$\Rightarrow \quad \frac{dr}{dt} = -\frac{Rk}{H}$$

This is the DE representing the variation in the radius of the water surface with time. The inital radius is R and the final radius is 0. If the time taken for the entire water to evaporate is T, we have

$$\int_R^0 dr = -\frac{Rk}{H}\int_0^T dt$$

$$\Rightarrow \quad T = \frac{H}{k}$$

Note that the time taken is independent of the radius of the cone and depends only on its height. Thus, for example, two cones full of water, with the same height, but one of them having a radius say a 1000 times larger than the other, will become empty in the same amount of time! ◀

Example – 4

A curve C has the property that if the tangent drawn at any point P on C meets the coordinate axes at A and B, then P is the mid-point of AB. C passes through (1, 1). Determine its equation.

Solution: Let the curve be $y = f(x)$.

The tangent at any point $P(x, y)$ has the equation

$$Y - y = \frac{dy}{dx}(X - x)$$

This meets the axes in $A\left(x - y\frac{dx}{dy}, 0\right)$ and $B\left(0, y - x\frac{dy}{dx}\right)$. Since P itself is the mid-point of AB, we have

$$x - y\frac{dx}{dy} = 2x, y - x\frac{dy}{dx} = 2y$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x}$$

This is in *V.S* form and can be solved by straight forward integration:

$$\int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \ln y = -\ln x + \ln k \qquad (k \text{ is an arbitrary constant})$$

$$\Rightarrow \quad xy = k$$

Since the curve C passes through (1, 1), we have $k = 1$. Thus, the equation of C is

$$xy = 1$$

◀

Example – 5

A and B are two separate reservoirs of water. The capacity of reservoir A is double the capacity of reservoir B. Both the reservoirs are filled completely with water, their inlets are closed and then water is released simultaneuously from both the reservoirs. The rate of flow of water out of each reservoir at any instant of time is proportinal to the quantity of water in the reservoir at that time. One hour after the water is released, the quantity of water in reservoir A is $1\frac{1}{2}$ times the quantity of water in reservoir B. After how many hours do both the reservoirs have the same quantity of water?

Solution: Assume the intial volumes of water in A and B to be 2 V and V.

Denote the volume of water in A and B by v_1 and v_2 respectively. We have,

$$\frac{dv_1}{dt} = -k_A v_1, \quad \frac{dv_2}{dt} = -k_B v_2$$

where k_A and k_B are constants of proportionality (not given). These two DEs are in VS form and the solution can be obtained by simple integration.

$$\int_{2V}^{v_1(t)} \frac{dv_1}{v_1} = \int_0^t -k_A dt, \qquad \int_V^{v_2(t)} \frac{dv_2}{v_2} = \int_0^t -k_B \, dt$$

$$\Rightarrow \quad \ln \frac{v_1(t)}{2V} = -k_A t, \qquad \ln \frac{v_2(t)}{V} = -k_B t$$

$$\Rightarrow \quad v_1(t) = 2Ve^{-k_A t}, \qquad v_2(t) = Ve^{-k_B t}$$

It is given that

$$v_1(t=1) = \frac{3}{2} v_2(t=1)$$

$$\Rightarrow \quad 2Ve^{-k_A} = \frac{3}{2} Ve^{k_B}$$

$$\Rightarrow \quad e^{(k_B - k_A)} = \frac{3}{4}$$

$$\Rightarrow \quad k_B - k_A = \ln\left(\frac{3}{4}\right)$$

Let T be the time at which the volumes in the two reservoirs become equal. We thus have,

$$v_1(t=T) = v_2(t=T)$$

$$\Rightarrow \quad 2Ve^{-k_A T} = Ve^{-k_B T}$$

$$\Rightarrow \quad e^{(k_A - k_B)T} = 2$$

$$\Rightarrow \quad (k_A - k_B)T = \ln 2$$

$$\Rightarrow \quad T \ln\left(\frac{4}{3}\right) = \ln 2$$

$$\Rightarrow \quad T = \frac{\ln 2}{\ln(4/3)} \text{ hours}$$

◀

ASSIGNMENT

[LEVEL - I]

Q. 1 Show that

(a) $\frac{d^2y}{dx^2}=0$ represents all non-vertical lines.

(b) $\frac{d^2x}{dy^2}=0$ represents all non-horizontal lines.

Q. 2 Find the DE corresponding to parabolas whose axis of symmetry is parallel to the x-axis.

Q. 3 Find the DE corresponding to the curve $y = a\sin(mx + b)$.

Q. 4 Solve the DE $(x+y+1)dx+(3x+3y)dy=0$

Q. 5 Solve the DE $x\sin\left(\frac{y}{x}\right)dy=\left\{y\sin\left(\frac{y}{x}\right)-x\right\}dx$

Q. 6 Solve the DE $\left(x^2-4xy-2y^2\right)dx+\left(y^2-4xy-2x^2\right)dy=0$

Q. 7 Solve the DE $\frac{dy}{dx}=\cos(x+y)+\sin(x+y)$

Q. 8 Solve the DE $\left(x+2y^3\right)dy=ydx$

Q. 9 Solve the DE $\frac{dy}{dx}+\frac{y}{x}\ln x=\frac{y}{x^2}(\ln x)^2$

Q. 10 Show that the curve for which the normal at every point passes through a fixed point is a circle.

[LEVEL - II]

Q 1. Find the general solution of the differential equation $\frac{dy}{dx}=\left(\frac{x+2y-3}{2x+y+3}\right)^2$

Q. 2 Let $u(x)$ and $v(x)$ satisfy the differential equations

$$\frac{du}{dx}+p(x)\cdot u=f(x)\text{ and }\frac{dv}{dx}+p(x)\cdot v=g(x)$$

respectively where $p(x)$, $f(x)$ and $g(x)$ are continuous functions. If $u(x_1)>v(x_1)$ for some x_1 and $f(x)>g(x)$ for all $x>x_1$, prove that any point (x,y), where $x > x_1$, does not satisfy the equations $y=u(x)$ and $y=v(x)$.

Q. 3 Determine the equation of the curve passing through the origin in the form $y = f(x)$, which satisfies the differential equation $\frac{dy}{dx} = \sin(10x + 6y)$.

Q. 4 A normal is drawn at a point $P(x, y)$ of a curve, It meets the x-axis at Q. If PQ is of constant length k, then show that the differential equation describing such curves is

$$y\frac{dy}{dx} = \pm\sqrt{k^2 - y^2}$$

and find the equation of such a curve passing through $(0, k)$

Q. 5 A curve passing through the point (1, 1) has the property that the perpendicular distance of the normal at any point P on the curve from the origin is equal to the distance of P from the x-axis. Determine the equation to the curve.

Q. 6 A hemispherical tank of radius 2 meters is initially full of water and has an outlet of 12 cm^2 cross-sectional area at the bottom. The outlet is opened at some instant. The flow through the output is according to the law $v(t) = 0.6\sqrt{2gh(t)}$, where $v(t)$ and $h(t)$ are respectively the velocity of the flow through the outlet and the height of water level above the outlet at time t and g the acceleration due to gravity. Find the time it takes to empty the tank.

Q. 7 A country has a food deficit of 10%. Its population grows continuously at a rate of 3% per year. Its annual food production every year is 4% more than that of the last year. Assuming that the average food requirement per person remains constant, prove that the country will become self-sufficient in food after n years, where n is the smallest integer bigger than or equal to

$$\frac{\ln 10 - \ln 9}{\ln(1.04) - 0.03}$$

Q. 8 A curve $f(x)$ passes through the point (0, 1). A curve $g(x) = \int_{-\infty}^{x} f(x)\,dx$ passes through the point $(0, 1/n)$. The tangents drawn to both the curves at points having the same abscissa, intersect on the x-axis. Find $f(x)$.

Q. 9 Given the curves $y = f(x)$ passing through the point (0, 1) and $y = \int_{-\infty}^{x} f(t)\,dt$ passing through the point $\left(0, \frac{1}{2}\right)$. The tangents drawn to both the curves at the points with equal abscissa intersect on the x-axis. Find the curve $y = f(x)$.

Q. 10 Solve the DE $(1 + x^2)\,dy = y(x - y)\,dx$

ASSIGNMENT

(ANSWERS)

[LEVEL - I]

2. $\left[\frac{d^3x}{dy^3}=0\right]$

3. $\left[y\frac{d^3y}{dx^3}=\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}\right]$

4. $\left[x+3y+\frac{3}{2}\ln\left|2x+2y-1\right|=C\right]$

5. $\left[\ln x-\cos\frac{y}{x}=C\right]$

6. $\left[\frac{x^3}{3}+\frac{y^3}{3}-2x^2y-2xy^2=C\right]$

7. $\left[\ln\left|1+\tan\left(\frac{x+y}{2}\right)\right|=x+C\right]$

8. $\left[\frac{x}{y}=y^2+C\right]$

9. $\left[\frac{1}{x\ln y}=\frac{1}{2x^2}+C\right]$

[LEVEL - II]

1. $\left[(x+3)^3-(y-3)^3=C(x-y+6)^4\right]$

3. $\left[y=\frac{1}{3}\left\{\tan^{-1}\left(\frac{5\tan 4x}{4-3\tan 4x}\right)-5x\right\}\right]$

4. $\left[x^2+y^2=k^2\right]$

5. $\left[x^2+y^2=2x\right]$

6. $\left[T=\frac{7\pi\times 10^4}{135\sqrt{g}}\sec\right]$

8. $\left[f(x)=e^{nx}\right]$

9. $\left[f(x)=e^{2x}\right]$

10. $\left[\frac{\sqrt{1+x^2}}{y}=\ln\left(x+\sqrt{1+x^2}\right)+C\right]$

ANSWERS

TRY YOURSELF - I

1. $\left[(x-y)^2\left(1+\left(\frac{dy}{dx}\right)^2\right)=\left(x+y\frac{dy}{dx}\right)^2\right]$

2. $\left[\left(x^2-y\right)^2\frac{dy}{dx}=2xy\right]$

3. $\left[yy'=x\left(y'\right)^2+xyy''\right]$

TRY YOURSELF - II

1. $\left[\ln\left(x^2+y^2\right)=2ay+C\right]$

2. $\left[x^2\left(y^2+2xy\right)=C\right]$

3. $\left[(y+1)^2+(x+2)(y+1)-(x+2)^2=C\right]$

4. $\left[y\sin x=\frac{1}{2}\sin^2 x+C\right]$

5. $\left[\frac{x^2}{y^2}=-x^2\ln x+\frac{x^2}{2}+C\right]$