## Section - 3 NORMALS

Many properties of normals in parabolas are non-trivial and important enough to be discussed independently of tangents. This is what we do here. Once again, we use the parabola $y^2 = 4ax$ for illustration purposes although the discussion can easily be generalised.

**EQUATION OF NORMAL AT $P(x_1, y_1)$** :

Let $P(x_1, y_1)$ be a point on the parabola; thus $y_1^2 = 4ax_1$.
The slope of the tangent at *P* is

$$m_T = \left.\frac{dy}{dx}\right|_{(x_1, y_1)} = \frac{2a}{y_1}$$

Thus, the slope of the normal is

$$m_N = \frac{-y_1}{2a}$$

The equation of the normal at $(x_1, y_1)$ can now be written using point-slope form:

$$\boxed{y - y_1 = \frac{-y_1}{2a}(x - x_1)}$$

**EQUATION OF NORMAL AT $P(at^2, 2at)$** :

The point *P* is now given in parametric form. Using the equation of the normal derived above, we can write the equation in parametric form by using $(at^2, 2at)$ instead of $(x_1, y_1)$ :

$$y - 2at = -\frac{2at}{2a}(x - at^2)$$

$$\Rightarrow \quad \boxed{y + tx = 2at + at^3}$$

Note that this normal has slope $m = -t$. Thus, the same equation can also be specified in terms of slope as described below.

**EQUATION OF NORMAL WITH SLOPE *m*** :

Instead of *t*, we use $-m$ in the equation above :

$$y - mx = -2am - am^3$$

$$\Rightarrow \quad \boxed{y = mx - 2am - am^3}$$

Note that this is the normal at the point $(at^2, 2at)$ which in terms of $m$ is $(am^2, -2am)$.

The cubic equation in *m* hints that from a given point *P*, **three** normals can be drawn to the parabola. Let us try to prove this. Suppose that the point *P* is (*h*, *k*). Since the normal (s) of slope *m* passes through *P*, we have

$$k = mh - 2am - am^3$$

$$\Rightarrow \quad am^3 + (2a - h)m - k = 0 \qquad ...(1)$$

This gives three values of $m$, say $m_1$, $m_2$ and $m_3$ and thus three corresponding normals. However, the roots $m_1$, $m_2$ and $m_3$ may not all be real. Two of them could be imaginary (one will always be real). Thus, depending on the coefficients in(1), we could either have **one** or **three** normals from *P* to the parabola. (There is also the possibility of two identical roots in which case only two normals will actually exist)

In case there are three normals, these will intersect the parabola at $(am_i^2, -2am_i)$ for *i* = 1, 2 3. The sum of the ordinates of these points is

$$-2a(m_1 + m_2 + m_3)$$

which is 0 from (1).

## Example – 26

What are the points on the parabola $y^2 = 4ax$ from which three distinct normals can be drawn to the parabola?

**Solution:** Assume a point $P(at^2, 2at)$ from which three normal to the parabola can be drawn. We basically need to find the range of the variable *t* for which this is possible. The equation of an arbitrary normal to this parabola can be written as

$$y = mx - 2am - am^3$$

If this passes through *P*, we have

$$2at = at^2 m - 2am - am^3$$

$$\Rightarrow \quad m^3 + (2 - t^2)m + 2t = 0$$

$$\Rightarrow \quad (m + t)(m^2 - mt + 2) = 0$$

One root for *m* is –*t* which actually gives the normal at *P* itself. This should have been expected because a normal to the parabola itself on any point can obviously always be drawn.

The other two roots for *m* are real and distinct if

$$t^2 > 8$$

This is the condition that the parameter *t* must satisfy if we are to be able to draw three real and distinct normals from *P* to the parabola. ◀

## Example – 27

A normal is drawn to the parabola $y^2 = 4ax$ at the point *t*. Find the other point at which this normal intersects the parabola.

**Solution:** The equation of the normal at $t$ is

$$y + tx = 2at + at^3 \qquad ...(1)$$

If this intersects the parabola again at $t_1$ then $(at_1^2, 2at_1)$ must satisfy (1). Thus,

$$2at_1 + at_1^2 t = 2at + at^3$$

$$\Rightarrow \quad at(t_1^2 - t^2) = 2a(t - t_1)$$

$$\Rightarrow \quad t(t_1 + t) = -2$$

$$\Rightarrow \quad \boxed{t_1 = -t - \frac{2}{t}}$$

This very frequently used result tells us how to find the other end point of the normal chord which is normal at a point $t$ to $y^2 = 4ax$. ◀

Sometimes, a question might be posed pertaining to normals. Instead of solving it entirely from scratch, we could use already known properties that we might have discussed earlier, say, in the section on tangents. This saves a lot of time in a subject like co-ordinate geometry.

For example, suppose $AB$ is a focal chord of a parabola. What is the angle between the tangent at $A$ and the normal at $B$?

We discussed earlier that tangents at extremities of any focal chord are perpendicular. Thus, a tangent at one end-point and a normal at the other must be parallel !

Thus, you can see that the skill you need to master for this subject is to remember certain well known and frequently used results and use them to your advantage as much as possible.

**Example – 28**

The normals at $t_1$ and $t_2$ of the parabola $y^2 = 4ax$ meet at $t_3$. Prove that $t_1 t_2 = 2$.

**Solution:** We can very easily solve this question using the result of the last example. Since the normal drawn at $t_1$ intersects the parabola again in $t_3$, we have $t_3 = -t_1 - \frac{2}{t_1}$. Similarly, $t_3 = -t_2 - \frac{2}{t_2}$. Comparing the two gives $t_1 t_2 = 2$. But let us solve it without using this result. The equation of the normals at $t_1$ and $t_2$ are

$$y + t_1 x = 2at_1 + at_1^3 \qquad ...(1)$$

$$y + t_2 x = 2at_2 + at_2^3 \qquad ...(2)$$

Both of these must be satisfied by the point $t_3$, i.e., by $(at_3^2, 2at_3)$. Thus,

$$2at_3 + at_1 t_3^2 = 2at_1 + at_1^3 \qquad ...(3)$$

$$2at_3 + at_2 t_3^2 = 2at_2 + at_2^3 \qquad ...(4)$$

Subtracting(4) from (3), we have

$$at_3^3(t_1-t_2)=2a(t_1-t_2)+a(t_1^3-t_2^3)$$

$$\Rightarrow \quad t_3^2=2+t_1^2+t_2^2+t_1t_2 \qquad ...(5)$$

Using (5) in (3), we finally have

$$4a^2(2+t_1^2+t_2^2+t_1t_2)=(at_1t_2^2+at_1^2t_2)^2$$

$$\Rightarrow \quad 8+4\{(t_1+t_2)^2-t_1t_2\}=t_1^2t_2^2(t_1+t_2)^2$$

$$\Rightarrow \quad (t_1t_2-2)\{(t_1+t_2)^2(t_1t_2+2)+4\}=0$$

which gives $t_1t_2=2,$ as required. ◀

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On page - 29, four important properties that tangents in any parabola satisfy have been listed. Since a normal at a point is perpendicular to the tangent at that point, we can use the properties of tangents to deduce the corresponding properties for normals.

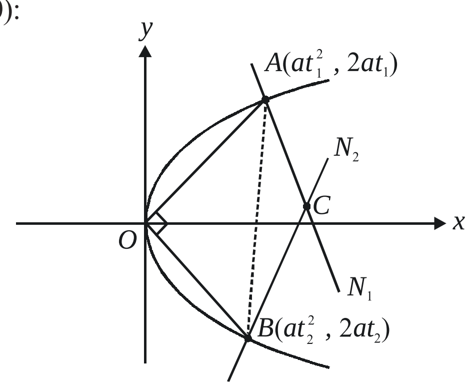
| | | | |
|---|---|---|---|
| * (A) | Tangent at any point bisects the angle $\theta$ between the focal chord through that point and the perpendicular to the directrix from that point | $\Rightarrow$ | Normal at any point bisects the **external** angle between the focal chord and the perpendicular to the directrix from that point, i.e. it bisects the supplementary angle $180-\theta.$ |
| * (B) | The tangent at one extremity of any focal chord of a parabola is perpendicular to the normal at the other extremity. | $\Rightarrow$ | The tangent at one extremity of any focal chord of a parabola is parallel to the normal at the other extremity. (We've already stated this earlier) |

Note that from property (A) above, we can also deduce that the normal at any point of a parabola is equally inclined to the focal chord through that point and the axis of the parabola. You are urged to prove this independently as an exercise.

**Example – 29**

Find the locus of the point of intersection of normals drawn to the parabola $y^2=4ax$ at the extremities of a chord which subtends a right angle at the vertex of the parabola.

**Solution:** Let $A(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$ be the extremities of a chord which subtends a right angle at the vertex (0, 0):




*AB* is a chord which subtends a right angle at *O*. The normals at *A* and *B*, $N_1$ and $N_2$, intersect at *C*. We need to find the locus of *C*.

Fig - 31

Since $OA \perp OB,$ we have

$$\underbrace{\left(\frac{2at_1 - 0}{at_1^2 - 0}\right)}_{\text{slope of } OA} \times \underbrace{\left(\frac{2at_2 - 0}{at_2^2 - 0}\right)}_{\text{slope of } OB} = -1$$

$$\Rightarrow \quad t_1 t_2 = -4$$

The equation to $N_1$ and $N_2$ can be written using the standard form of a normal at a point $t$ :

$$N_1 : y + t_1 x = 2at_1 + at_1^3$$

$$N_2 : y + t_2 x = 2at_2 + at_2^3$$

Let the intersection of $N_1$ and $N_2$ be the point $C(h, k)$. The co-ordinates of $C$ can be evaluated by solving the equations of $N_1$ and $N_2$ simultaneously :

$$h = 2a + a(t_1^2 + t_2^2 + t_1 t_2)$$

$$= -2a + a(t_1^2 + t_2^2)$$

and $$k = -at_1 t_2 (t_1 + t_2)$$

$$= 4a(t_1 + t_2)$$

We thus have, by eliminating $t_1$ and $t_2$, a relation in $h$ and $k$:

$$\frac{k^2}{16a^2} = \frac{h + 2a}{a} - 8$$

$$\Rightarrow \quad k^2 = 16a(h - 6a)$$

Using $(x, y)$ instead of $(h, k)$, the required locus is

$$y^2 = 16a(x - 6a)$$ ◀

**Example – 30**

Find the locus of the point of intersection of the three normals to the parabola $y^2 = 4ax,$ two of which are inclined at right angles to each other.

**Solution:** Let $P(h, k)$ be the point whose locus we wish to determine. Any normal to the given parabola can be written as

$$y = mx - 2am - am^3$$

If this passes through $P(h, k)$, we have

$$k = mh - 2am - am^3 \qquad ...(1)$$

Let $m_1$, $m_2$ and $m_3$ be the three roots of this cubic. It is given that two of the normals are perpendicular, implying that the product of two of these three slopes, say $m_1$ and $m_2$ is $-1$, i.e. $m_1 m_2 = -1$.

From (1), we have

$$m_1 m_2 m_3 = \frac{-k}{a}$$

$$\Rightarrow \qquad m_3 = \frac{k}{a}$$

Substituting this value of $m_3$ back in (1), we obtain a relation between $h$ and $k$ :

$$k = \frac{hk}{a} - 2k - \frac{ak^3}{a^3}$$

$$\Rightarrow \qquad k(k^2 + (3a - h)a) = 0$$

Using $(x, y)$ instead of $(h, k)$ the required locus is

$$y(y^2 + (3a - x)a) = 0$$ ◀

## TRY YOURSELF - III

**Q. 1** Find the equation of the normal to $y^2 = 4x$ which is perpendicular to $2x + 6y + 5 = 0$.

**Q. 2** The normal at any point *P* on the parabola $y^2 = 4ax$ meets its axis in *A* and the *y*-axis in *B*. Let *O* be the origin. The rectangle *OACB* is completed. Find the locus of *C*.

**Q. 3** Prove that the normal to $y^2 = 4ax$ at a non-zero point whose *x* and *y* coordinates are equal, subtends a right angle at the focus.

**Q. 4** Prove that the normal at any point to a parabola is equally inclined to the focal chord passing through that point and the axis of the parabola.

**Q. 5** Have you ever heard of parabolic mirrors being very effective in concentrating incident light energy at a particular a point ? Can you think of a reason ? What would that particular point be ?