Complex Algebra

1. If $(x+iy) = \sqrt{\frac{a+ib}{c+id}}$ then $(x^2+y^2)^2 =$

(1) $\frac{a^2+b^2}{c^2+d^2}$ (2) $\frac{a+b}{c+d}$ (3) $\frac{c^2+d^2}{a^2+b^2}$ (4) $\left(\frac{a^2+b^2}{c^2+d^2}\right)^2$

Soln. $x+iy = \sqrt{\frac{a+ib}{c+id}}$ on squaring

$\Rightarrow x^2-y^2+2xyi = \frac{a+ib}{c+id}$

Taking modulus both the sides, we get

$\sqrt{(x^2-y^2)^2+(2xy)^2} = \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$ $\because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Squaring again,

$(x^2)^2+(y^2)^2-2x^2y^2+4x^2y^2 = \frac{a^2+b^2}{c^2+d^2}$

$\Rightarrow (x^2)^2+(y^2)^2+2x^2y^2 = \frac{a^2+b^2}{c^2+d^2}$

$\Rightarrow (x^2+y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$ Ans.

Method 2 Short cut method

$$x+iy = \sqrt{\frac{a+ib}{c+id}} \quad \text{—(i)}$$

Replace i by $-i$

$$x-iy = \sqrt{\frac{a-ib}{c-id}} \quad \text{—(ii)}$$

Multiplying (i) and (ii), we get

$$x^2+y^2 = \sqrt{\frac{(a+ib)(a-ib)}{(c+id)(c-id)}}$$

Squaring $\Rightarrow (x^2+y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$

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2. If $\sqrt{a+ib} = x+iy$, then possible value of $\sqrt{a-ib}$ is

(1) x^2+y^2 (2) $\sqrt{x^2+y^2}$ (3) $x+iy$ (4) $x-iy$

Soln. $\because \sqrt{a+ib} = x+iy$

Squaring both the sides.

$$a+ib = x^2-y^2+2xyi$$

Taking conjugate both the sides.

$$a-ib = x^2-y^2-2xyi = (x-iy)^2$$

$$\sqrt{a-ib} = x-iy$$

3. The number of non-zero integral solution of the equation $|1-i|^x = 2^x$ is

(1) Infinite (2) 1 (3) 2 (4) None of these

Soln. $|1-i|^x = 2^x \Rightarrow \left(\sqrt{(1)^2+(-1)^2}\right)^x = 2^x \Rightarrow (\sqrt{2})^x = 2^x$

$$\Rightarrow 2^{x/2} = 2^x \Rightarrow x = 0$$

$\Rightarrow$ No non zero integral Solution.

4. If $z = re^{i\theta}$, then $|e^{iz}| =$

(1) $e^{r\sin\theta}$ (2) $e^{-r\sin\theta}$ (3) $e^{-r\cos\theta}$ (4) $e^{r\cos\theta}$

Soln. $\because z = re^{i\theta} = r(\cos\theta + i\sin\theta)$

$\therefore iz = ri\cos\theta - r\sin\theta$

$$\therefore e^{iz} = e^{-r\sin\theta + ri\cos\theta} = e^{-r\sin\theta}\, e^{(r\cos\theta)i}$$

$$|e^{iz}| = |e^{-r\cos\theta}|\ |e^{(r\cos\theta)i}|$$

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$\therefore |e^{iz}| = e^{-r\sin\theta} \times 1$ $\left[\because |e^{i\theta}| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1\right]$

$= e^{-r\sin\theta}$

5. If $-1 + \sqrt{-3} = re^{i\theta}$, then θ is equal to

(1) $\frac{\pi}{3}$ (2) $-\frac{\pi}{3}$ (3) $\frac{2\pi}{3}$ (4) $-\frac{2\pi}{3}$

Soln. $-1 + \sqrt{3}i = r\cos\theta + ir\sin\theta$ $\left[\because e^{i\theta} = \cos\theta + i\sin\theta\right]$

$r\cos\theta = -1$

$r\sin\theta = \sqrt{3}$ $\therefore \tan\theta = \frac{\sqrt{3}}{-1}$ $\therefore \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

[$\because$ z lies in II quadrant]

6. Real Part of $e^{e^{i\theta}}$ is

(1) $e^{\cos\theta}[\cos(\sin\theta)]$ (2) $e^{\cos\theta}[\cos(\cos\theta)]$

(3) $e^{\sin\theta}[\sin(\cos\theta)]$ (4) $e^{\sin\theta}[\sin(\sin\theta)]$

Soln. $e^{i\theta} = \cos\theta + i\sin\theta$

$\therefore e^{e^{i\theta}} = e^{\cos\theta + i\sin\theta} = e^{\cos\theta} e^{i\sin\theta}$

$= e^{\cos\theta}[\cos(\sin\theta) + i\sin(\sin\theta)]$

$\therefore$ Real part of $e^{e^{i\theta}} = e^{\cos\theta}[\cos(\sin\theta)]$

7. If $z = \frac{1 + i\sqrt{3}}{\sqrt{3} + i}$, then $(\bar{z})^{100}$ lies in

(i) 1st quadrant (ii) II quadrant

(iii) 3rd quadrant (iv) IV quadrant

Soln. $\bar{z} = \frac{1 - i\sqrt{3}}{\sqrt{3} - i} = \frac{\frac{1}{2} - i\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2} - \frac{1}{2}i} = \frac{\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}}{\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}}$

$= \frac{e^{-i\pi/3}}{e^{-i\pi/6}} = e^{-\pi/6\, i} = \cos\frac{\pi}{6} - i\sin\frac{\pi}{6}$

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$(z)^{100} = \left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^{100}$

$= \left(\cos\frac{100\pi}{6} - i\sin\frac{100\pi}{6}\right)$ by Demoivre's Theorem

$= \cos\frac{50\pi}{3} - i\sin\frac{50\pi}{3}$

$= \cos\left(17\pi - \frac{\pi}{3}\right) - i\sin\left(17\pi - \frac{\pi}{3}\right)$

$= -\cos\frac{\pi}{3} - i\sin\frac{\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

which is lies in III quadrant.

8. If $x + \frac{1}{x} = \sqrt{3}$, then $x =$

(a) $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$
(b) $\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$
(c) $\sin\frac{\pi}{6} + i\cos\frac{\pi}{6}$
(d) None.

Soln. $\frac{x^2+1}{x} = \sqrt{3} \Rightarrow x^2 - \sqrt{3}x + 1 = 0$

$\therefore x = \frac{\sqrt{3} \pm \sqrt{3-4}}{2} = \frac{\sqrt{3} \pm i}{2} = \frac{\sqrt{3}}{2} \pm \frac{1}{2}i$

$= \cos\frac{\pi}{6} \pm i\sin\frac{\pi}{6}$

9. The imaginary part of $\tan^{-1}\left(\frac{5i}{3}\right)$ is

(1) 0 (2) ∞ (3) $\log 2$ (4) $\log 4$

Soln. $\tan^{-1}\left(\frac{5i}{3}\right) = i\tan^{-1}\left(\frac{5}{3}\right) = \frac{i}{2}\log\left(\frac{\frac{5}{3}+1}{\frac{5}{3}-1}\right) = \frac{i}{2}\log 4$

$= \frac{i}{2} 2\log 2 = i\log 2$

$\therefore$ imaginary part of $\tan^{-1}\left(\frac{5i}{3}\right) = \log 2$

10. If $z = \frac{7-i}{3-4i}$ then $z^{14} =$

(1) 2^7 (2) $2^7 i$ (3) $2^{14} i$ (4) $-2^7 i$

Soln: $z = \frac{7-i}{3-4i} \times \frac{3+4i}{3+4i} = \frac{21+4+28i-3i}{25} = \frac{25+25i}{25}$

$= 1 + i$

$z^2 = (1+i)^2 = 1 + i^2 + 2i = 1 - 1 + 2i = 2i$

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$\therefore \quad z^{14} = (z^2)^7 = (2i)^7 = 2^7 i^7 = -2^7 i$

11. Length of the line segment joining the points $-1-i$ and $2+3i$ is

(1) -5 (2) 15 (3) 5 (4) 25

Soln. length of line segment joining two affix z_1 and z_2 is $|z_2 - z_1|$

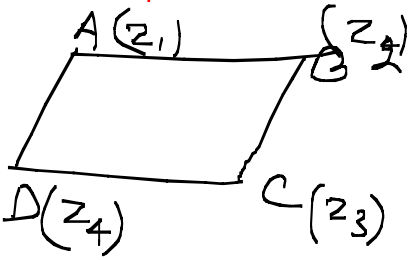
$\therefore$ Required length $= |2+3i+1+i| = |3+4i| = 5$

12. The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order. if and only if

(1) $z_1 + z_4 = z_2 + z_3$ (2) $z_1 + z_3 = z_2 + z_4$

(3) $z_1 + z_2 = z_3 + z_4$ (4) None of these

Soln. $\because$ diagonals of a parallelogram bisect each other



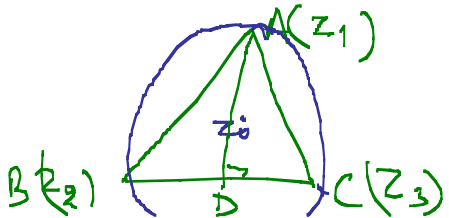
$\therefore$ mid point of AC = mid point of BD

$\Rightarrow \quad \frac{z_1+z_3}{2} = \frac{z_2+z_4}{2} \Rightarrow z_1+z_3 = z_2+z_4$

13. Let the complex numbers z_1, z_2 and z_3 be the vertices of an equilateral triangle. Let z_0 be the circumcentre of the triangle, $z_1^2 + z_2^2 + z_3^2 =$

(1) z_0^2 (2) $-z_0^2$ (3) $3z_0^2$ (4) $-3z_0^2$

Soln. To solve such kind of questions students are advice to take



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Suitable choice of axes. Let BC is on x axis. and AD is ⊥ on BC, where D is mid point of BC.

In equilateral triangle Circum Centre, Centroid and Incentre coincides. Let AB = BC = CA = 2a.

∴ D is at origin ∴ Coordinate of B and C are respectively (−a, 0) and (a, 0) respectively.

$$AD = \sqrt{AC^2 - DC^2} = \sqrt{(2a)^2 - (a)^2} = \sqrt{3a^2} = \sqrt{3}\,a.$$

∴ Coordinate of A will be $(0, \sqrt{3}a)$

Hence, Coordinate of Centroid z_0 is given by

$$\left(\frac{a + 0 + -a}{3}, \frac{0 + \sqrt{3}a + 0}{3}\right) \text{ i.e } z_0\left(0, \frac{a}{\sqrt{3}}\right)$$

$\therefore z_1 = 0 + \sqrt{3}a\,i$ $\quad \therefore z_1^2 = -3a^2$

$z_2 = -a + 0 \cdot i$ $\quad \therefore z_2^2 = a^2$

$z_3 = a + 0 \cdot i$ $\quad \therefore z_3^2 = a^2$

$z_0 = 0 + \frac{a}{\sqrt{3}}i$ $\quad z_0^2 = -\frac{a^2}{3} \Rightarrow 3z_0^2 = -a^2$ —(1)

Also, $z_1^2 + z_2^2 + z_3^2 = -3a^2 + a^2 + a^2 = -a^2 = 3z_0^2$

Proved.

14. If three Complex numbers are in A.P. then they lie on

(I) A Circle in the Complex plane.

(II) A Straight line in the Complex plane.

(III) A parabola in a Complex plane

(IV) None of these.

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Soln $\because z_1, z_2, z_3$ are in A.P

$\therefore z_2 - z_1 = z_3 - z_2 \Rightarrow 2z_2 = z_1 + z_3$.

$\Rightarrow z_2 = \frac{z_1 + z_3}{2}$

i.e z_2 is midpoint of line joining z_1 and z_2.

$\therefore z_1, z_2, z_3$ are collinear.

15. If a and b are real numbers between 0 and 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle, then

(1) $a = b = 2 + \sqrt{3}$ (2) $a = b = 2 - \sqrt{3}$

(3) $a = 2 - \sqrt{3}$, $b = 2 + \sqrt{3}$ (4) none of these

Soln. $\because z_1, z_2, z_3$ form an equilateral triangle.

$\therefore |z_1 - z_2| = |z_2 - z_3| = |z_1 - z_3|$

$\Rightarrow |a - 1 + (1-b)i| = |1 + bi| = |a + i|$

$\Rightarrow \sqrt{(a-1)^2 + (1-b)^2} = \sqrt{1 + b^2} = \sqrt{a^2 + 1}$

$\Rightarrow a^2 + b^2 - 2a - 2b + 2 = 1 + b^2 = a^2 + 1$

(I) (II) (III)

from (I) and (III) $a^2 = b^2 \Rightarrow a = \pm b$

$\therefore a^2 + b^2 - 2a - 2b + 2 = 1 + b^2$

$\Rightarrow a^2 + a^2 + 2a \pm 2a + 2 = 1 + a^2$

$\Rightarrow a^2 + 2a \pm 2a + 1 = 0$

$\Rightarrow a^2 + 2a - 2a + 1 = 0$

$\Rightarrow a^2 + 1 = 0$

which is not possible

$a^2 + 2a + 2a + 1 = 0$

$a^2 + 4a + 1 = 0$

$a = \frac{-4 \pm \sqrt{16 - 4}}{2}$

$= \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$

i.e $a = b = -2 \pm \sqrt{3}$

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16. In the Argand plane, the vector $z = 4 - 3i$ is turning the clockwise sense through 180° and stretched three times. The complex Number represented by the new vector is

(1) $12 + 9i$ (2) $12 - 9i$

(3) $-12 - 9i$ (4) $-12 + 9i$

Soln. When a complex number turn in clockwise sense through 180°. it means it lies in opposite quadrant Hence it would be obtained by multiplying by i^2 i.e -1

Hence Complex no would be $z_1 = -4 + 3i$

Since it Streched three times, Hence new Complex number will be $z_2 = 3z_1$

$= -12 + 9i$

17. The Centre of a Regular Polygon of n sides is located at the point $Z = 0$ and one of its vertex z_1 is known. If z_2 be the vertex adjacent to z_1, then z_2 is equal to

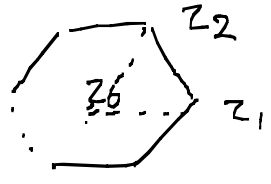
(I) $z_1\left(\cos\frac{2\pi}{n} \pm i \sin\frac{2\pi}{n}\right)$

(II) $z_1\left(\cos\frac{\pi}{n} \pm i \sin\frac{\pi}{n}\right)$

(III) $z_1\left(\cos\frac{\pi}{2n} \pm i \sin\frac{\pi}{2n}\right)$

(IV) None of these

Soln: $\because$ z_2 is the the next adjacent vertex of regular polygon of n sides.



Then angle at Centre will be $\frac{2\pi}{n}$. it means the z_2 is a Complex number obtained by Rotation of an angle $\frac{2\pi}{n}$ about z_0 i.e origin. Hence,

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z_2 will be $z_1 e^{i\theta}$ or $z_1 e^{-i\theta}$ accordingly in anticlockwise or clockwise sense of rotation respectively.

$\therefore z_2 = z_1 e^{\pm i\theta}$ $\quad \because \theta = \frac{2\pi}{n}$.

$= z_1 \left(\cos\frac{2\pi}{n} \pm i \sin\frac{2\pi}{n} \right)$

18. The equation $z\bar{z} + (2-3i)z + (2+3i)\bar{z} + 4 = 0$ represent a circle of radius

(1) 2 (2) 3 (3) 4 (4) 6

Soln: $\because$ $z\bar{z} + \bar{a}z + a\bar{z} + b = 0$ is the equation of circle with radius $= \sqrt{|a|^2 - b}$

$\therefore$ radius $= \sqrt{13-4} = \sqrt{9} = 3$

19. If $\frac{|z-2|}{|z-3|} = 2$ represent a circle, then its radius is equal to

(1) 1 (2) $1/3$ (3) $\frac{3}{4}$ (4) $2/3$

Soln: $z = x+iy$

$\therefore \frac{|z-2|}{|z-3|} = 2 \Rightarrow |z-2| = 2|z-3|$

$\Rightarrow |x+iy-2| = 2|x+iy-3|$

$\Rightarrow \sqrt{(x-2)^2+y^2} = 2\sqrt{(x-3)^2+y^2}$

$\Rightarrow (x-2)^2 + y^2 = 4[(x-3)^2+y^2]$

$\Rightarrow x^2+y^2-4x+4 = 4x^2+4y^2-24x+36$

$\Rightarrow 3x^2+3y^2-20x+32 = 0$

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$$\Rightarrow x^2+y^2-\frac{20}{3}x+\frac{32}{3}=0$$

[$\because$ If $x^2+y^2+2gx+2fy+c=0$ is a circle, then radius $=\sqrt{g^2+f^2-c}$]

$$\therefore \text{Radius} = \sqrt{\left(\frac{10}{3}\right)^2+(0)^2-\frac{32}{3}} = \sqrt{\frac{100}{9}-\frac{96}{9}}$$

$$= \sqrt{\frac{4}{9}} = \frac{2}{3}$$

20. If Centre of a regular hexagon is at origin and one of its vertex on argand diagram is $1+2i$, then its perimeter is

(1) $2\sqrt{5}$ (2) $6\sqrt{2}$ (3) $4\sqrt{5}$ (4) $6\sqrt{5}$

Soln. length of one of side of regular hexagon

$$= |z_0-z_1| = |1+2i| = \sqrt{1^2+2^2} = \sqrt{5}$$

Perimeter of regular hexagon $= 6a = 6\sqrt{5}$

21. A circle whose radius is r and Centre z_0. Then the equation of the circle is

(1) $z\bar{z} - z\bar{z}_0 - \bar{z}z_0 + z_0\bar{z}_0 = r^2$

(2) $z\bar{z} + z\bar{z}_0 - \bar{z}z_0 + z_0\bar{z}_0 = r^2$

(3) $z\bar{z} - z\bar{z}_0 + \bar{z}z_0 - z_0\bar{z}_0 = r^2$

(4) None of these

Soln. Eqn of circle will be $|z-z_0| = r$

$$\Rightarrow |z-z_0|^2 = r^2 \Rightarrow (z-z_0)(\bar{z}-\bar{z}_0) = r^2$$

$$\Rightarrow (z-z_0)(\bar{z}-\bar{z}_0) = r^2$$

$$\Rightarrow z\bar{z} - z\bar{z}_0 - \bar{z}z_0 + z_0\bar{z}_0 = r^2$$

[$\because z\bar{z} = |z|^2$, $\overline{z_1-z_2} = \bar{z}_1-\bar{z}_2$]

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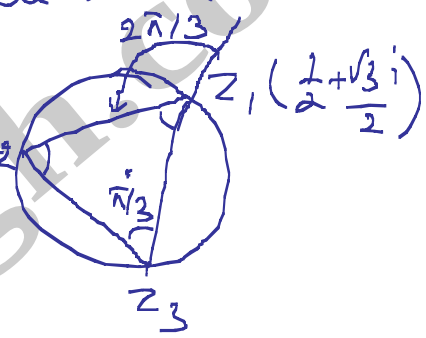
22. Let z_1, z_2, z_3 be three vertices of an equilateral $\triangle$ circumscribing the circle $|z| = \frac{1}{2}$. If $z_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and z_1, z_2, z_3 are in anticlockwise sense then z_2 is

(1) $1+\sqrt{3}i$ (2) $1-\sqrt{3}i$ (3) 1 (4) -1

Soln: $\because |z| = \frac{1}{2}$ is the equation of circle whose centre at origin and radius $\frac{1}{2}$.



$\therefore z_2$ is the vertex obtained by rotation of an angle $\pi/3$ of z_1.

$\therefore z_2 = z_1 e^{i\pi/3}$

$= \left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) e^{i2\pi/3} = e^{i\pi/3} e^{2\pi/3 i} = e^{i\pi}$.

$= (\cos\pi + i\sin\pi) = -1$ Ans.

23. For all complex number z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, The minimum value of $|z_1 - z_2|$ is

(1) 0 (2) 2 (3) 7 (4) 17

Soln: $\because |z_1| = 12$ is the equation of circle whose centre at origin and radius is 12.

$|z_2 - 3 - 4i| = 5$ is also equation of circle whose centre at $(3,4)$ and radius 5.

it means second circle lies inside other.

as the distance of centre from origin $= \sqrt{3^2+4^2} = 5$

and radius is also 5. i.e second circle must be passes through origin.

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