SOLVED EXAMPLES

Example – 1

Let F_1 and F_2 be the foci of an ellipse with eccentricity e. For any point P on the ellipse, prove that

$$\left(\frac{\tan \angle PF_1F_2}{2}\right)\tan\left(\frac{\angle PF_2F_1}{2}\right)=\frac{1-e}{1+e}$$

Solution: Assume the ellipse to be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$ and let P be the point $(a\cos\theta,\ b\sin\theta).$

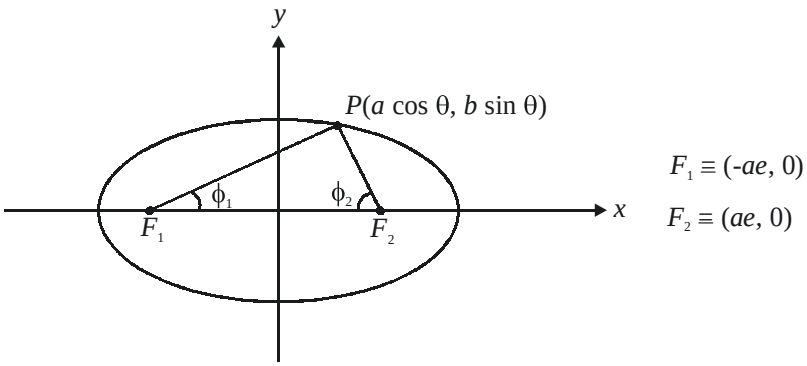


Fig - 38

Using the sine rule in $\Delta PF_1F_2,$ we have

$$\frac{PF_2}{\sin\phi_1}=\frac{PF_1}{\sin\phi_2}=\frac{F_1F_2}{\sin(\pi-(\phi_1+\phi_2))}$$

$$\Rightarrow \quad \frac{PF_2+PF_1}{\sin\phi_1+\sin\phi_2}=\frac{F_1F_2}{\sin(\phi_1+\phi_2)}$$

$$\Rightarrow \quad \frac{2a}{\sin\phi_1+\sin\phi_2}=\frac{2ae}{\sin(\phi_1+\phi_2)} \qquad \left\{\begin{array}{l}\because\ PF_1+PF_2 \text{ will always equal}\\ \text{the length of the major axis}\end{array}\right\}$$

$$\Rightarrow \quad \frac{\cos\left(\frac{\phi_1+\phi_2}{2}\right)}{\cos\left(\frac{\phi_1-\phi_2}{2}\right)}=e$$

$$\Rightarrow \quad \frac{\cos\left(\frac{\phi_1-\phi_2}{2}\right)-\cos\left(\frac{\phi_1+\phi_2}{2}\right)}{\cos\left(\frac{\phi_1-\phi_2}{2}\right)+\cos\left(\frac{\phi_1+\phi_2}{2}\right)}=\frac{1-e}{1+e}$$

$$\Rightarrow \quad \tan\left(\frac{\phi_1}{2}\right)\tan\left(\frac{\phi_2}{2}\right)=\frac{1-e}{1+e}$$

This is the desired result. ◀

Example – 2

Let d be the perpendicular distance from the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to the tangent drawn at a point P on the ellipse. If F_1 and F_2 are the two foci of the ellipse, prove that

$$(PF_1 - PF_2)^2 = 4a^2\left(1-\frac{b^2}{d^2}\right)$$

Solution: Let P be the point $(a\cos\theta, b\sin\theta)$ whereas F_1 and F_2 are given by $(\pm ae, 0)$.

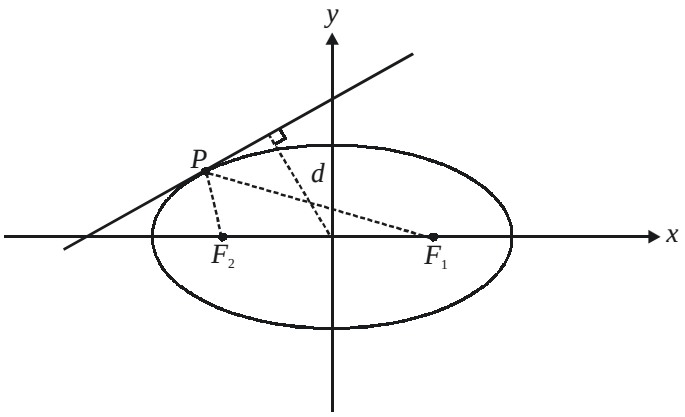


Fig - 39

By definition, the focal distance of any point on an ellipse is e times the distance of that point from the corresponding directrix. Thus,

$$PF_1 = \left|e\left(a\cos\theta - \frac{a}{e}\right)\right|$$

$$= |ae\cos\theta - a|$$

$$= a - ae\cos\theta$$

$$PF_2 = e\left(a\cos\theta + \frac{a}{e}\right)$$

$$= a + ae\cos\theta$$

$$\Rightarrow \quad (PF_1 - PF_2)^2 = 4a^2e^2\cos^2\theta \qquad ...(1)$$

Now, the equation of the tangent at P is

$$bx\cos\theta + ay\sin\theta - ab = 0$$

The distance of $(0, 0)$ from this tangent is d. Thus,

$$d = \frac{ab}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}}$$

$$\Rightarrow \quad \frac{b^2}{d^2} = \sin^2 \theta + \frac{b^2}{a^2} \cos^2 \theta$$

$$\Rightarrow \quad 1 - \frac{b^2}{d^2} = \left(1 - \frac{b^2}{a^2}\right) \cos^2 \theta$$

$$= e^2 \cos^2 \theta$$

$$\Rightarrow \quad 4a^2 \left(1 - \frac{b^2}{d^2}\right) = 4a^2 e^2 \cos^2 \theta \qquad \ldots(2)$$

From (1) and (2), we see that the equality stated in the question does indeed hold. ◀

Example – 3

Find the radius of the largest circle with centre (1, 0) that can be inscribed inside the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$.

Solution: The following diagram shows the largest such circle. Observe it carefully :

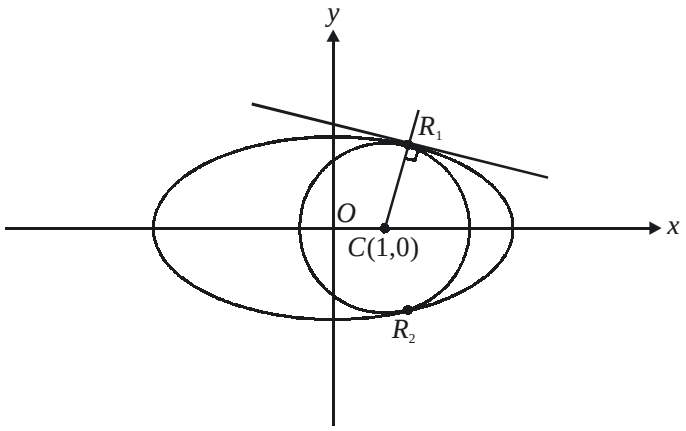


Fig - 40

Note that the largest possible circle lying completely inside the ellipse must touch it, say, at the points R_1 and R_2, as shown. At these points, it will be possible to draw common tangents to the circle and the ellipse.

Let point R_1 be $(4\cos\theta,\ 2\sin\theta)$. The equation of the tangent at R_1 is

$$\frac{x \cos\theta}{4} + \frac{y \sin\theta}{2} = 1$$

$$\Rightarrow \quad x \cos\theta + 2y \sin\theta = 4$$

If CR_1 is perpendicular to this tangent (which must happen if this tangent is to be common to both the ellipse and the circle), we have

$$\frac{2\sin\theta - 0}{4\cos\theta - 1} \times \frac{-\cos\theta}{2\sin\theta} = -1$$

$$\Rightarrow \quad \cos\theta = \frac{1}{3}$$

Thus, R_1 is $\left(\frac{4}{3}, \frac{4\sqrt{2}}{3}\right)$. The largest possible radius is therefore

$$r_{max} = CR_1 = \sqrt{\left(\frac{4}{3} - 1\right)^2 + \left(\frac{4\sqrt{2}}{3} - 0\right)^2}$$

$$= \sqrt{\frac{11}{3}}$$

◀

Example – 4

A tangent is drawn to the ellipse $E_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which cuts the ellipse $E_2 : \frac{x^2}{c^2} + \frac{y^2}{d^2} = 1$ at the points A and B. Tangents to this second ellipse at A and B intersect at right angles. Prove that

$$\frac{a^2}{c^2} + \frac{b^2}{d^2} = 1$$

Solution: Let the point of intersection of the two tangents be $P(h, k)$.

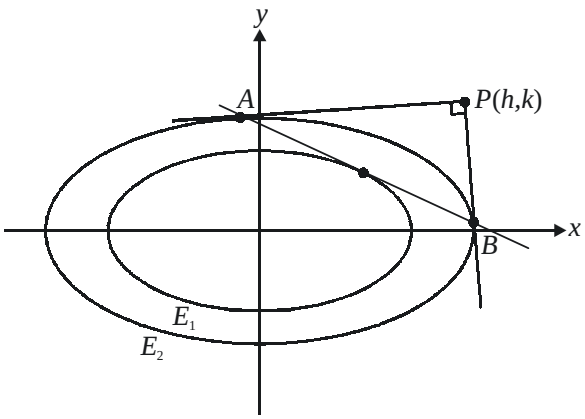


Fig - 41

Note that since AB is the chord of contact for the tangents drawn from P to E_2, we have the equation of AB as

$$T(h, k) = 0$$

$$\Rightarrow \quad \frac{hx}{c^2} + \frac{ky}{d^2} = 1$$

$$\Rightarrow \quad y = \left(\frac{-d^2 h}{c^2 k}\right)x + \frac{d^2}{k}$$

If AB is to touch the inner ellipse E_1, the condition of tangency must be satisfied :

$$\frac{d^4}{k^2} = a^2 \cdot \frac{d^4h^2}{c^4k^2} + b^2$$

$$\Rightarrow \quad c^4d^4 = a^2d^4h^2 + b^2c^4k^2 \qquad ...(1)$$

Since PA and PB intersect at right angles, P must lie on the director circle of the ellipse E_2. Thus,

$$h^2 + k^2 = c^2 + d^2 \qquad ...(2)$$

(1) and (2) can be considered a system of equations in the variables h^2 and k^2 :

$$(a^2d^4)h^2 + (b^2c^4)k^2 = c^4d^4$$

$$h^2 + \qquad k^2 = c^2 + d^2$$

If these relations are to hold for variable h and k, they must in fact be identical. Thus, these variables can now easily be eliminated to obtain :

$$\frac{a^2d^4}{c^4d^4} = \frac{1}{c^2+d^2} \quad ; \quad \frac{b^2c^4}{c^4d^4} = \frac{1}{c^2+d^2}$$

$$\Rightarrow \quad \frac{a^2}{c^2} = \frac{c^2}{c^2+d^2} \quad ; \quad \frac{b^2}{d^2} = \frac{d^2}{c^2+d^2}$$

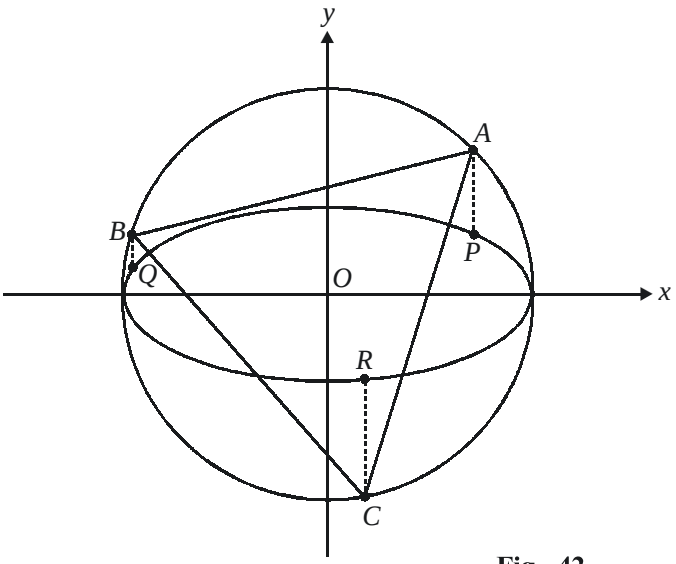
$$\Rightarrow \quad \frac{a^2}{c^2} + \frac{b^2}{d^2} = 1$$

◀

Example – 5

Let ABC be an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$. Perpendiculars from A, B, C to the major axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (where $a > b$) meet the ellipse respectively at P, Q, R so that P, Q and R lie on the same side of the major axis of A, B and C respectively. Prove that the normals to the ellipse at P, Q and R are concurrent.

Solution: The following figure graphically portrays the situation described in the question :



We need to show that the normals at P, Q and R are concurrent

Fig - 42

All the information given about ABC being equilateral and all can be reduced to this single piece of significant information : the polar angles of A, B and C, and hence, the eccentric angles of P, Q and R, will be evenly spaced at $\frac{2\pi}{3}$, by virtue of ΔABC being equilateral.

Thus, we can assume the eccentric angels of P, Q and R to be $\theta,\ \theta+\frac{2\pi}{3},\ \theta-\frac{2\pi}{3}$.

Now, the equation of a normal to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at eccentric angle ϕ is given by

$$ax\sec\phi - by\cos ec\phi = a^2-b^2.$$

$$\Rightarrow \qquad ax\sin\phi - by\cos\phi = \frac{(a^2-b^2)\sin 2\phi}{2}.$$

Thus, the normals at P, Q and R are respectively given by

$$N_P \quad : \quad ax\sin\theta - by\cos\theta = \frac{(a^2-b^2)\sin 2\theta}{2}$$

$$N_Q \quad : \quad ax\sin\left(\theta+\frac{2\pi}{3}\right) - by\cos\left(\theta+\frac{2\pi}{3}\right) = \frac{\left(a^2-b^2\right)}{2}\sin\left(2\theta+\frac{4\pi}{3}\right)$$

$$N_R \quad : \quad ax\sin\left(\theta-\frac{2\pi}{3}\right) - by\cos\left(\theta-\frac{2\pi}{3}\right) = \frac{(a^2-b^2)}{2}\sin\left(2\theta-\frac{4\pi}{3}\right)$$

Let us evaluate Δ, the determinant of the coefficients of these three equations :

$$\Delta = \frac{ab(a^2-b^2)}{2}\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

Using the row operation $R_1 \to R_1+R_2+R_3$, the first row reduces to zero, which means that

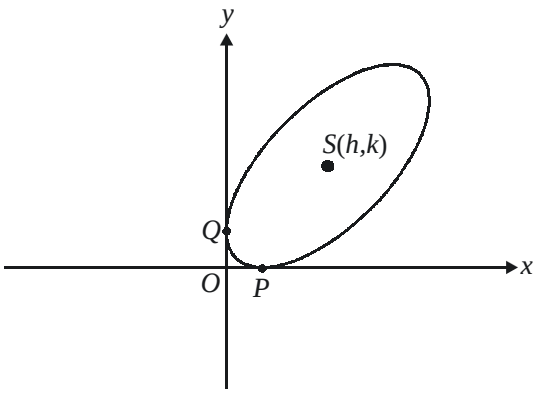
$$\Delta = 0$$

Thus, the normals at P, Q and R must be concurrent. ◀

Example – 6

An ellipse slides between two lines at right angles to one another. Show that the locus of its centre is a circle.

Solution: It would be easiest to assume the two lines to be the coordinate axis, and an ellipse of fixed dimensions sliding between these two lines as shown below :



The ellipse has fixed dimensions, say,
Major axis = $2a$
Minor axis = $2b$
Assume the centre S to be (h, k)

Fig - 43

From the view-point of the ellipse, since the tangents to it at P and Q intersect at right angles at O, the point O must lie on the director circle of the ellipse. Since the radius of the director circle is $\sqrt{a^2+b^2}$, we must have

$$OS = \sqrt{a^2+b^2}$$

$$\Rightarrow \quad OS^2 = a^2+b^2$$

$$\Rightarrow \quad h^2+k^2 = a^2+b^2$$

This must be the locus of the centre S! It can be written in $x-y$ form as

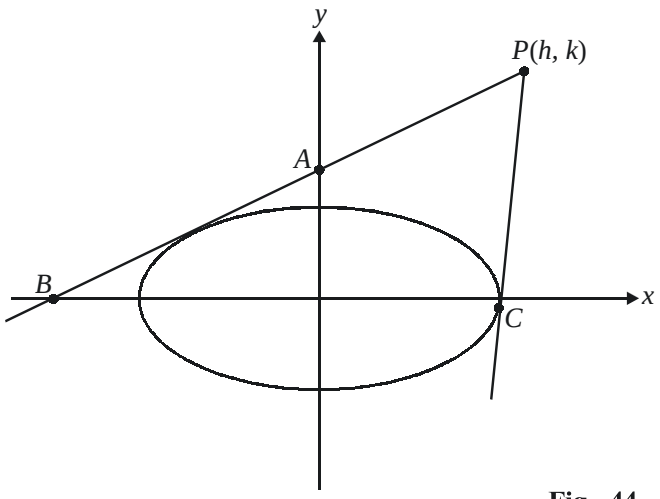
$$x^2+y^2 = a^2+b^2$$

It is evident that this is a circle centred at the origin and of radius $\sqrt{a^2+b^2}$. ◀

Example – 7

Find the locus of the point P such that tangents drawn from it to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meet the coordinates axes in concyclic points.

Solution:



The tangents from $P(h, k)$ to the ellipse meet the coordinate axes in A, B, C and D (not shown) which are concyclic.

Fig - 44

The pair of tangents PA and PC has the joint equation

$$J_1 \quad : \quad T^2(h,k) = S(x,y)S(h,k)$$

$$\Rightarrow \quad J_1 \quad : \quad \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)-\left(\frac{hx}{a^2}+\frac{ky}{b^2}-1\right)^2 = 0$$

The coordinate axes has the joint equation

$$J_2 \quad : \quad xy = 0$$

We can treat J_1 and J_2 as two curves, which intersect in four different points A, B, C, D. Any second degree curve through these four points can be written in terms of a parameter λ as

$$J_1 + \lambda J_2 = 0$$

We now simply find that λ for which this represents a circle, since A, B, C, D are given to be concyclic.

$$\Rightarrow \quad \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)-\left(\frac{hx}{a^2}+\frac{ky}{b^2}-1\right)^2 + \lambda xy = 0$$

This represents a circle if

$$\textbf{Coeff. of } x^2 = \textbf{Coeff. of } y^2 \quad \Rightarrow \quad \frac{k^2}{a^2b^2}-\frac{1}{a^2} = \frac{h^2}{a^2b^2}-\frac{1}{b^2}$$

$$\Rightarrow \quad h^2 - k^2 = a^2 - b^2 \qquad ...(1)$$

$$\textbf{Coeff. of } xy = 0 \quad \Rightarrow \quad \lambda = \frac{2hk}{a^2b^2} \qquad ...(2)$$

(1) itself gives the locus of $P(h, k)$ as

$$x^2 - y^2 = a^2 - b^2$$

Example – 8

Through any arbitrary fixed point $P(\theta)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$ chords at right angles are drawn, such that the line joining the extremities of these chords meets the normal through P at the point Q. Prove that Q is fixed for all such chords.

Solution:

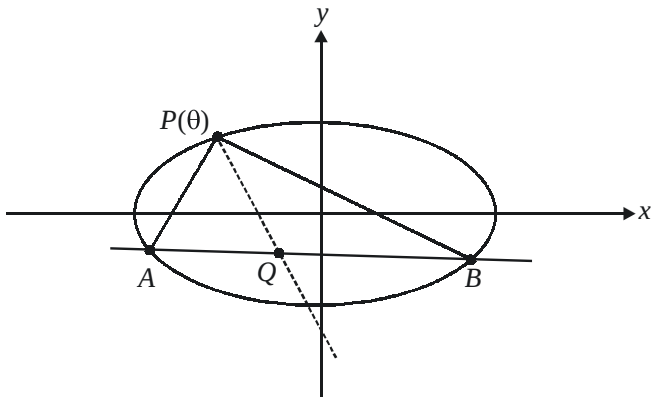


Fig - 45

We can assume the eccentric angles of A and B as θ_1 and θ_2.

The normal at P has the equation :

$$PQ: ax\sec\theta - by\,\text{cosec}\,\theta = a^2 - b^2 \qquad ...(1)$$

The chord AB has the equation

$$AB: \frac{x}{a}\cos\left(\frac{\theta_1+\theta_2}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)=\cos\left(\frac{\theta_1-\theta_2}{2}\right) \qquad ...(2)$$

Also, since $PA \perp PB,$ we have

$$\underbrace{\frac{-b\cos\left(\frac{\theta+\theta_1}{2}\right)}{a\sin\left(\frac{\theta+\theta_1}{2}\right)}}_{\text{slope of chord } PA} \times \underbrace{\frac{-b\cos\left(\frac{\theta+\theta_2}{2}\right)}{a\sin\left(\frac{\theta+\theta_1}{2}\right)}}_{\text{slope of chord } PB} = -1$$

$$\Rightarrow \quad a^2\sin\left(\frac{\theta+\theta_1}{2}\right)\sin\left(\frac{\theta+\theta_2}{2}\right)+b^2\cos\left(\frac{\theta+\theta_1}{2}\right)\cos\left(\frac{\theta+\theta_2}{2}\right)=0$$

Using trignometric formulae, this expression can be rearranged to

$$\cos\left(\theta+\frac{\theta_1+\theta_2}{2}\right)=\frac{a^2+b^2}{a^2-b^2}\cos\left(\frac{\theta_1-\theta_2}{2}\right) \qquad ...(3)$$

From (2) and (3), we have

$$AB: \frac{x}{a}\cos\left(\frac{\theta_1+\theta_2}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)=\frac{a^2-b^2}{a^2+b^2}\cos\left(\theta+\frac{\theta_1+\theta_2}{2}\right) \quad ...(4)$$

The point Q can now be obtained as the intersection of the lines represented by (1) and (4). Let us write them as a system and solve for Q using the Cramer's rule:

$$PQ: \quad (a\sec\theta)x+(-b\,\text{cosec}\,\theta)y+(b^2-a^2)=0$$

$$AB: \quad (b\cos\alpha)x+(a\sin\alpha)y-\frac{ab(a^2-b^2)}{a^2+b^2}\cos(\theta+\alpha)=0$$

where $\alpha=\frac{\theta_1+\theta_2}{2}$ has been substituted for convenience.

We now have

$$\frac{x}{\frac{ab^2(a^2-b^2)}{a^2+b^2}\text{cosec}\,\theta\cos(\theta+\alpha)-a\sin\alpha(b^2-a^2)}=\frac{y}{b(b^2-a^2)\cos\alpha+\frac{a^2b(a^2-b^2)}{a^2+b^2}\sec\theta\cos(\theta+\alpha)}$$

$$=\frac{1}{a^2\sec\theta\sin\alpha+b^2\text{cosec}\,\theta\cos\alpha}$$

$$\Rightarrow \qquad x=\frac{\frac{ab^2(a^2-b^2)}{a^2+b^2}\cos(\theta+\alpha)-a\sin\theta\sin\alpha(b^2-a^2)}{\sin\theta(a^2\sec\theta\sin\alpha+b^2\text{cosec}\,\theta\cos\alpha)}$$

$$=\frac{a(a^2-b^2)}{a^2+b^2}\cos\theta \quad ...(5)$$

and $$y=\frac{b(b^2-a^2)\cos\alpha\cos\theta+\frac{a^2b(a^2-b^2)}{a^2+b^2}\cos(\theta+\alpha)}{\cos\theta(a^2\sec\theta\sin\alpha+b^2\text{cosec}\,\theta\cos\alpha)}$$

$$=\frac{b(b^2-a^2)\sin\theta}{a^2+b^2} \quad ...(6)$$

Thus, the point Q, whose x and y coordinates are given by (5) and (6) respectively, can be seen to be independent of α or $\frac{\theta_1+\theta_2}{2}$. Q is the therefore fixed for such pairs of chords PA and PB and depends only on the eccentric angle of P. ◀

Example – 9

Consider three points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $P(\theta_1)$, $Q(\theta_2)$ and $R(\theta_3)$. What is the area of ΔPQR ? When is this area maximum ?

Solution: The three points have the coordinates

$$P \equiv (a\cos\theta_1,\ b\sin\theta_1);\ Q \equiv (a\cos\theta_2, b\sin\theta_2);\ R \equiv (a\cos\theta_3, b\sin\theta_3)$$

The area of this triangle, by the determinant formula, is

$$\Delta = \frac{1}{2}\begin{vmatrix} a\cos\theta_1 & b\sin\theta_1 & 1 \\ a\cos\theta_2 & b\sin\theta_2 & 1 \\ a\cos\theta_3 & b\sin\theta_3 & 1 \end{vmatrix} \qquad \ldots(1)$$

$$= \frac{ab}{2}\left\{\cos\theta_1(\sin\theta_2-\sin\theta_3)+\sin\theta_1(\cos\theta_3-\cos\theta_2)+(\cos\theta_2\sin\theta_3-\sin\theta_2\cos\theta_3)\right\}$$

$$= \frac{ab}{2}\left\{\sin(\theta_2-\theta_1)+\sin(\theta_1-\theta_3)+\sin(\theta_3-\theta_2)\right\}$$

$$= \frac{ab}{2}\left\{\sin(\theta_2-\theta_1)+2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\theta_1+\theta_2}{2}-\theta_3\right)\right\}$$

$$= ab\sin\left(\frac{\theta_1-\theta_2}{2}\right)\left\{\cos\left(\theta_3-\frac{\theta_1+\theta_2}{2}\right)-\cos\left(\frac{\theta_1-\theta_2}{2}\right)\right\}$$

$$= 2ab\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_2-\theta_3}{2}\right)\sin\left(\frac{\theta_3-\theta_1}{2}\right)$$

This is the area of the triangle *PQR*.

To find its maximum value, we use a rather indirect route. Suppose we had to calculate the area Δ' of a triangle inscribed in the circle $x^2+y^2=a^2$ with the same polar angles as *P*, *Q*, *R*. The only difference between Δ' and Δ will be that in the determinant expression for Δ in (1), we will have all '*a*' instead of '*b*' in the terms of the second column.

This means that Δ and Δ' will always be in a constant ratio :

$$\frac{\Delta}{\Delta'}=\frac{b}{a}$$

Thus, the maximum for Δ will be achieved in the same configuration as the one in which the maximum of Δ' be achieved !

Since the area of a triangle inscribed in a circle has the maximum value when that triangle is equilateral (this should be intuitively obvious but can also be easily proved), Δ' and hence Δ will be maximum when

$$|\theta_1 - \theta_2| = |\theta_2 - \theta_3| = |\theta_3 - \theta_1| = \frac{2\pi}{3}$$

Thus, the three eccentric angels must be equally spaced apart at $\frac{2\pi}{3}$. ◀

Example – 10

Prove that the circle on any focal distance as diameter touches the auxiliary circle of the ellipse.

Solution: Let $P(\theta)$ be an arbitrary point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and let F_1 be one of its foci.

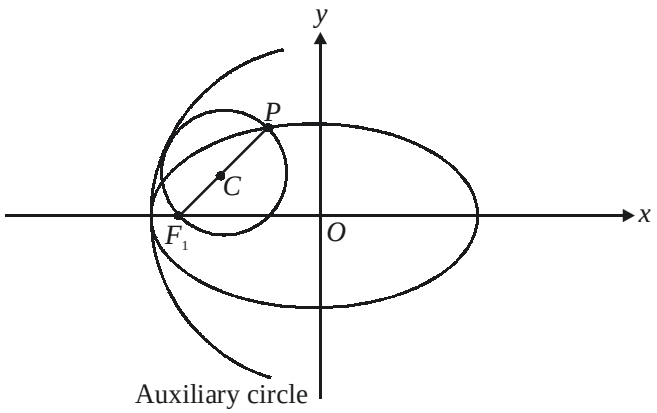


Fig - 46

The radius of the auxiliary is *a*. The circle on PF_1 as diameter will touch the auxiliary circle (internally) if:

$$OC + (\text{radius of this circle}) = a$$

C, being the mid-point of PF_1, has the coordinates

$$C \equiv \left(\frac{a\cos\theta - ae}{2}, \frac{b\sin\theta}{2} \right)$$

Thus,

$$OC = \sqrt{\left(\frac{a\cos\theta - ae}{2}\right)^2 + \left(\frac{b\sin\theta}{2}\right)^2}$$

$$= \frac{1}{2}\sqrt{a^2\cos^2\theta + b^2\sin^2\theta + a^2e^2 - 2a^2e\cos\theta}$$

$$= \frac{a}{2}\sqrt{\cos^2\theta + (1-e^2)\sin^2\theta + e^2 - 2e\cos\theta}$$

$$= \frac{a}{2}\sqrt{1 + e^2\cos^2\theta - 2e\cos\theta}$$

$$= \frac{a}{2}(1 - e\cos\theta)$$

Also, the radius of the inner circle is

$$CF_1 \equiv \sqrt{\left(\frac{a\cos\theta - ae}{2} + ae\right)^2 + \left(\frac{b\sin\theta}{2}\right)^2}$$

$$= \frac{a}{2}\sqrt{\cos^2\theta + (1-e^2)\sin^2\theta + e^2 + 2\cos\theta}$$

$$= \frac{a}{2}(1 + e\cos\theta)$$

This gives

$$OC + CF_1 = a$$

which proves the stated assertion. ◀

ASSIGNMENT

[LEVEL - I]

Q 1. If the normals at the points θ_1, θ_2 and θ_3 on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are concurrent, prove that

$$\begin{vmatrix} \sec\theta_1 & \text{cosec}\theta_1 & 1 \\ \sec\theta_2 & \text{cosec}\theta_2 & 1 \\ \sec\theta_3 & \text{cosec}\theta_3 & 1 \end{vmatrix} = 0$$

Q 2. The tangent and normal at a point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meet the minor axis at A and B respectively. Prove that the circle with AB as diameter passes through P as well as the two foci of the ellipse

Q 3. Let AB be a focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ The tangent at A and the normal at B intersect in P. Find the locus of P.

Q 4. A line intersects the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at A and B and the parabola $y^2=4d(x+a)$ at C and D. AB subtends a right angle at the centre of the ellipse. Tangents to the parabola at C and D intersect in E. Prove that the locus of E is

$$y^2+4d^2=4d^2(x+2a)^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$$

Q 5. Prove that the common chords of an ellipse and a circle are equally inclined to the axes of the ellipse.

Q 6. Prove that the minimum length of the intercept made by the axes on the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is equal to $a + b$.

Q 7. If d is the length of the perpendicular from the focus F of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ upon a tangent at any point P on the ellipse, prove that

$$1+\frac{b^2}{d^2}=\frac{2a}{FP}$$

Q 8. With a given point and line as focus and directrix, a series of ellipses are described. Prove that the locus of the extremities of their minor axis is a parabola.

Q 9. The tangent at any point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets its auxiliary circle in A and B. AB subtends a right angle at the centre of the ellipse. Let e be the eccentricity of the ellipse and θ be the eccentric angle of P. Show that

$$\sin\theta=\sqrt{\frac{1}{e^2}-1}$$

Q 10 Show that the common tangent to the ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2x}{c} \text{ and } \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{2x}{c}=0$$

subtends a right angle at the origin.

[LEVEL - II]

Q 11. If a variable point P on an ellipse is joined to the two foci F_1 and F_2, show that the incentre of ΔPF_1F_2 lies on another ellipse.

Q 12. Two tangents to a given ellipse intersect at right angles. Prove that the sum of the squares of the chords which the auxiliary circle intercepts on these tangents, is a constant.

Q 13. From any point P on the ellipse $E_1 : a^2x^2 + b^2y^2 = (a^2 + b^2)^2$, tangents are drawn to the ellipse $E_2 : b^2x^2 + a^2y^2 = a^2b^2$, which touch E_2 at A and B. Prove that the orthocentre of ΔPAB lies on E_2.

Q 14. Normals to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the points $(x_i, y_i), i = 1, 2, 3,$ are concurrent. Show that

$$\begin{vmatrix} x_1 & y_1 & x_1y_1 \\ x_2 & y_2 & x_2y_2 \\ x_3 & y_3 & x_3y_3 \end{vmatrix} = 0$$

Q 15. A point P moves so that circle with PQ (where $Q \equiv (a, 0)$) as diameter touches the circle $x^2 + y^2 = 4a^2$ internally. Prove that the locus of P is an ellipse.

Q 16. Normals at the points $\theta_1, \theta_2, \theta_3$ and θ_4 on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are concurrent. Prove that $(\cos\theta_1 + \cos\theta_2 + \cos\theta_3 + \cos\theta_4)\ (\sec\theta_1 + \sec\theta_2 + \sec\theta_3 + \sec\theta_4) = 4$

Q 17. Tangents are drawn from any point on $\frac{x^2}{16} + \frac{y^2}{9} = 1$ to the circle $x^2 + y^2 = 1.$ Find the locus of the mid-point of the chord of contact.

Q 18. A parallelogram circumscribes the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and two of its opposite vertices lie on the straight lines $x = \pm c.$ Find the locus of the other two vertices of the parallelogram.

Q 19. The base of a variable triangle is of a fixed length d and the sum of its side, s, is also fixed. Prove that the locus of the incentre of the triangle is an ellipse.

Q 20. Let $P(\theta)$ be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ A parabola is drawn having its focus at P and passing through the foci of the given ellipse. Show that two such parabolas can be drawn. Prove also that their directrices will be inclined at an angle 2θ.

ASSIGNMENT

(ANSWERS)

[LEVEL - I]

3. $\left[\frac{x^2}{a^2}+\frac{b^2y^2}{(2a^2-b^2)^2}=1\right]$

[LEVEL - II]

17. $\left[\frac{x^2}{16}+\frac{y^2}{9}=(x^2+y^2)^2\right]$

18. $\left[\frac{a^2y^2}{b^2c^2}=\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}\left(1-\frac{a^2}{c^2}\right)\right\}\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right\}\right]$

ANSWERS

TRY YOURSELF - I

1. $[abe]$

2. $\left[\frac{x^2}{9}+\frac{y^2}{25}=1\right]$

3. $\left[\pm\sqrt{3}-1,-2\right]$

4. $\left[5x^2+9y^2-54y+36=0\right]$

5. $\left[e=\frac{\sqrt{3}}{2}\right]$

TRY YOURSELF - II

2. $\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}=2\right]$

4. $\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}=\sec^2\alpha\right]$

5. $\left[x^2+y^2=a^2\right]$

8. $\left[\frac{x}{a}+\frac{y}{b}=\sqrt{2}\right]$

TRY YOURSELF - III

1. $\left[\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}\right]$

TRY YOURSELF - IV

3. $\left[-\frac{1}{\sqrt{5}},\cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)\right]$

5. $\left[4x+3y=12\right]$