

## SECTION 1

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:
  - Full Marks* : +4 If only (all) the correct option(s) is(are) chosen;
  - Partial Marks* : +3 If all the four options are correct but ONLY three options are chosen;
  - Partial Marks* : +2 If three or more options are correct but ONLY two options are chosen, both of which are correct;
  - Partial Marks* : +1 If two or more options are correct but ONLY one option is chosen and it is a correct option;
  - Zero Marks* : 0 If unanswered;
  - Negative Marks* : -2 In all other cases.
- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then
  - choosing ONLY (A), (B) and (D) will get +4 marks;
  - choosing ONLY (A) and (B) will get +2 marks;
  - choosing ONLY (A) and (D) will get +2 marks;
  - choosing ONLY (B) and (D) will get +2 marks;
  - choosing ONLY (A) will get +1 mark;
  - choosing ONLY (B) will get +1 mark;
  - choosing ONLY (D) will get +1 mark;
  - choosing no option(s) (i.e. the question is unanswered) will get 0 marks and
  - choosing any other option(s) will get -2 marks.

Q.1 Let

$$S_1 = \{(i, j, k) : i, j, k \in \{1, 2, \dots, 10\}\},$$

$$S_2 = \{(i, j) : 1 \leq i < j + 2 \leq 10, i, j \in \{1, 2, \dots, 10\}\},$$

$$S_3 = \{(i, j, k, l) : 1 \leq i < j < k < l, i, j, k, l \in \{1, 2, \dots, 10\}\}$$

and

$$S_4 = \{(i, j, k, l) : i, j, k \text{ and } l \text{ are distinct elements in } \{1, 2, \dots, 10\}\}.$$

If the total number of elements in the set  $S_r$  is  $n_r$ ,  $r = 1, 2, 3, 4$ , then which of the following statements is (are) **TRUE** ?

- (A)  $n_1 = 1000$       (B)  $n_2 = 44$       (C)  $n_3 = 220$       (D)  $\frac{n_4}{12} = 420$

Q.2 Consider a triangle  $PQR$  having sides of lengths  $p, q$  and  $r$  opposite to the angles  $P, Q$  and  $R$ , respectively. Then which of the following statements is (are) **TRUE** ?

(A)  $\cos P \geq 1 - \frac{p^2}{2qr}$

(B)  $\cos R \geq \left(\frac{q-r}{p+q}\right) \cos P + \left(\frac{p-r}{p+q}\right) \cos Q$

(C)  $\frac{q+r}{p} < 2 \sqrt{\frac{\sin Q \sin R}{\sin P}}$

(D) If  $p < q$  and  $p < r$ , then  $\cos Q > \frac{p}{r}$  and  $\cos R > \frac{p}{q}$

Q.3 Let  $f: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow \mathbb{R}$  be a continuous function such that

$$f(0) = 1 \text{ and } \int_0^{\frac{\pi}{3}} f(t) dt = 0$$

Then which of the following statements is (are) **TRUE** ?

(A) The equation  $f(x) - 3 \cos 3x = 0$  has at least one solution in  $\left(0, \frac{\pi}{3}\right)$

(B) The equation  $f(x) - 3 \sin 3x = -\frac{6}{\pi}$  has at least one solution in  $\left(0, \frac{\pi}{3}\right)$

(C)  $\lim_{x \rightarrow 0} \frac{x \int_0^x f(t) dt}{1 - e^{x^2}} = -1$

(D)  $\lim_{x \rightarrow 0} \frac{\sin x \int_0^x f(t) dt}{x^2} = -1$

- Q.4 For any real numbers  $\alpha$  and  $\beta$ , let  $y_{\alpha,\beta}(x)$ ,  $x \in \mathbb{R}$ , be the solution of the differential equation

$$\frac{dy}{dx} + \alpha y = x e^{\beta x}, \quad y(1) = 1.$$

Let  $S = \{y_{\alpha,\beta}(x) : \alpha, \beta \in \mathbb{R}\}$ . Then which of the following functions belong(s) to the set  $S$ ?

- (A)  $f(x) = \frac{x^2}{2} e^{-x} + \left(e - \frac{1}{2}\right) e^{-x}$
- (B)  $f(x) = -\frac{x^2}{2} e^{-x} + \left(e + \frac{1}{2}\right) e^{-x}$
- (C)  $f(x) = \frac{e^x}{2} \left(x - \frac{1}{2}\right) + \left(e - \frac{e^2}{4}\right) e^{-x}$
- (D)  $f(x) = \frac{e^x}{2} \left(\frac{1}{2} - x\right) + \left(e + \frac{e^2}{4}\right) e^{-x}$
- Q.5 Let  $O$  be the origin and  $\overrightarrow{OA} = 2\hat{i} + 2\hat{j} + \hat{k}$ ,  $\overrightarrow{OB} = \hat{i} - 2\hat{j} + 2\hat{k}$  and  $\overrightarrow{OC} = \frac{1}{2}(\overrightarrow{OB} - \lambda \overrightarrow{OA})$  for some  $\lambda > 0$ . If  $|\overrightarrow{OB} \times \overrightarrow{OC}| = \frac{9}{2}$ , then which of the following statements is (are) **TRUE**?

- (A) Projection of  $\overrightarrow{OC}$  on  $\overrightarrow{OA}$  is  $-\frac{3}{2}$
- (B) Area of the triangle  $OAB$  is  $\frac{9}{2}$
- (C) Area of the triangle  $ABC$  is  $\frac{9}{2}$
- (D) The acute angle between the diagonals of the parallelogram with adjacent sides  $\overrightarrow{OA}$  and  $\overrightarrow{OC}$  is  $\frac{\pi}{3}$

- Q.6 Let  $E$  denote the parabola  $y^2 = 8x$ . Let  $P = (-2, 4)$ , and let  $Q$  and  $Q'$  be two distinct points on  $E$  such that the lines  $PQ$  and  $PQ'$  are tangents to  $E$ . Let  $F$  be the focus of  $E$ . Then which of the following statements is (are) **TRUE** ?
- (A) The triangle  $PFQ$  is a right-angled triangle
  - (B) The triangle  $QPQ'$  is a right-angled triangle
  - (C) The distance between  $P$  and  $F$  is  $5\sqrt{2}$
  - (D)  $F$  lies on the line joining  $Q$  and  $Q'$

**SECTION 2**

- This section contains **THREE (03)** question stems.
- There are **TWO (02)** questions corresponding to each question stem.
- The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:  
*Full Marks* : +2 If **ONLY** the correct numerical value is entered at the designated place;  
*Zero Marks* : 0 In all other cases.

**Question Stem for Question Nos. 7 and 8****Question Stem**

Consider the region  $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \geq 0 \text{ and } y^2 \leq 4 - x\}$ . Let  $\mathcal{F}$  be the family of all circles that are contained in  $R$  and have centers on the  $x$ -axis. Let  $C$  be the circle that has largest radius among the circles in  $\mathcal{F}$ . Let  $(\alpha, \beta)$  be a point where the circle  $C$  meets the curve  $y^2 = 4 - x$ .

Q.7 The radius of the circle  $C$  is \_\_\_\_ .

Q.8 The value of  $\alpha$  is \_\_\_\_ .

### Question Stem for Question Nos. 9 and 10

#### Question Stem

Let  $f_1: (0, \infty) \rightarrow \mathbb{R}$  and  $f_2: (0, \infty) \rightarrow \mathbb{R}$  be defined by

$$f_1(x) = \int_0^x \prod_{j=1}^{21} (t-j)^j dt, \quad x > 0$$

and

$$f_2(x) = 98(x-1)^{50} - 600(x-1)^{49} + 2450, \quad x > 0,$$

where, for any positive integer  $n$  and real numbers  $a_1, a_2, \dots, a_n$ ,  $\prod_{i=1}^n a_i$  denotes the product of  $a_1, a_2, \dots, a_n$ . Let  $m_i$  and  $n_i$ , respectively, denote the number of points of local minima and the number of points of local maxima of function  $f_i$ ,  $i = 1, 2$ , in the interval  $(0, \infty)$ .

Q.9 The value of  $2m_1 + 3n_1 + m_1n_1$  is \_\_\_\_ .

Q.10 The value of  $6m_2 + 4n_2 + 8m_2n_2$  is \_\_\_\_ .

### Question Stem for Question Nos. 11 and 12

#### Question Stem

Let  $g_i: \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \rightarrow \mathbb{R}$ ,  $i = 1, 2$ , and  $f: \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \rightarrow \mathbb{R}$  be functions such that

$$g_1(x) = 1, g_2(x) = |4x - \pi| \text{ and } f(x) = \sin^2 x, \text{ for all } x \in \left[\frac{\pi}{8}, \frac{3\pi}{8}\right]$$

Define

$$S_i = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} f(x) \cdot g_i(x) dx, \quad i = 1, 2$$

Q.11 The value of  $\frac{16S_1}{\pi}$  is \_\_\_\_ .

Q.12 The value of  $\frac{48S_2}{\pi^2}$  is \_\_\_\_ .



## SECTION 3

- This section contains **TWO (02) paragraphs**. Based on each paragraph, there are **TWO (02)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:  
*Full Marks* : +3 If **ONLY** the correct option is chosen;  
*Zero Marks* : 0 If none of the options is chosen (i.e. the question is unanswered);  
*Negative Marks* : -1 In all other cases.

## Paragraph

Let

$$M = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 \leq r^2\},$$

where  $r > 0$ . Consider the geometric progression  $a_n = \frac{1}{2^{n-1}}$ ,  $n = 1, 2, 3, \dots$ . Let  $S_0 = 0$  and, for  $n \geq 1$ , let  $S_n$  denote the sum of the first  $n$  terms of this progression. For  $n \geq 1$ , let  $C_n$  denote the circle with center  $(S_{n-1}, 0)$  and radius  $a_n$ , and  $D_n$  denote the circle with center  $(S_{n-1}, S_{n-1})$  and radius  $a_n$ .

Q.13 Consider  $M$  with  $r = \frac{1025}{513}$ . Let  $k$  be the number of all those circles  $C_n$  that are inside  $M$ . Let  $l$  be the maximum possible number of circles among these  $k$  circles such that no two circles intersect. Then

(A)  $k + 2l = 22$     (B)  $2k + l = 26$     (C)  $2k + 3l = 34$     (D)  $3k + 2l = 40$

Q.14 Consider  $M$  with  $r = \frac{(2^{199}-1)\sqrt{2}}{2^{198}}$ . The number of all those circles  $D_n$  that are inside  $M$  is

(A) 198                      (B) 199                      (C) 200                      (D) 201

### Paragraph

Let  $\psi_1: [0, \infty) \rightarrow \mathbb{R}$ ,  $\psi_2: [0, \infty) \rightarrow \mathbb{R}$ ,  $f: [0, \infty) \rightarrow \mathbb{R}$  and  $g: [0, \infty) \rightarrow \mathbb{R}$  be functions such that  $f(0) = g(0) = 0$ ,

$$\psi_1(x) = e^{-x} + x, \quad x \geq 0,$$

$$\psi_2(x) = x^2 - 2x - 2e^{-x} + 2, \quad x \geq 0,$$

$$f(x) = \int_{-x}^x (|t| - t^2)e^{-t^2} dt, \quad x > 0$$

and

$$g(x) = \int_0^{x^2} \sqrt{t} e^{-t} dt, \quad x > 0.$$

Q.15 Which of the following statements is **TRUE** ?

(A)  $f(\sqrt{\ln 3}) + g(\sqrt{\ln 3}) = \frac{1}{3}$

(B) For every  $x > 1$ , there exists an  $\alpha \in (1, x)$  such that  $\psi_1(x) = 1 + \alpha x$

(C) For every  $x > 0$ , there exists a  $\beta \in (0, x)$  such that  $\psi_2(x) = 2x(\psi_1(\beta) - 1)$

(D)  $f$  is an increasing function on the interval  $\left[0, \frac{3}{2}\right]$

Q.16 Which of the following statements is **TRUE** ?

(A)  $\psi_1(x) \leq 1$ , for all  $x > 0$

(B)  $\psi_2(x) \leq 0$ , for all  $x > 0$

(C)  $f(x) \geq 1 - e^{-x^2} - \frac{2}{3}x^3 + \frac{2}{5}x^5$ , for all  $x \in \left(0, \frac{1}{2}\right)$

(D)  $g(x) \leq \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7$ , for all  $x \in \left(0, \frac{1}{2}\right)$

**SECTION 4**

- This section contains **THREE (03)** questions.
- The answer to each question is a **NON-NEGATIVE INTEGER**.
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:  
*Full Marks* : +4 If **ONLY** the correct integer is entered;  
*Zero Marks* : 0 In all other cases.

Q.17 A number is chosen at random from the set  $\{1, 2, 3, \dots, 2000\}$ . Let  $p$  be the probability that the chosen number is a multiple of 3 or a multiple of 7. Then the value of  $500p$  is \_\_\_ .

Q.18 Let  $E$  be the ellipse  $\frac{x^2}{16} + \frac{y^2}{9} = 1$ . For any three distinct points  $P, Q$  and  $Q'$  on  $E$ , let  $M(P, Q)$  be the mid-point of the line segment joining  $P$  and  $Q$ , and  $M(P, Q')$  be the mid-point of the line segment joining  $P$  and  $Q'$ . Then the maximum possible value of the distance between  $M(P, Q)$  and  $M(P, Q')$ , as  $P, Q$  and  $Q'$  vary on  $E$ , is \_\_\_ .

Q.19 For any real number  $x$ , let  $[x]$  denote the largest integer less than or equal to  $x$ . If

$$I = \int_0^{10} \left[ \sqrt{\frac{10x}{x+1}} \right] dx ,$$

then the value of  $9I$  is \_\_\_ .

**END OF THE QUESTION PAPER**